Nonlinear Analysis: Real World Applications 27 (2016) 1–25

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

A remark on removable singularity for nonlinear convection–diffusion equation

Lu GuoFu[a,b,*], Jiang Liang[a]

[a] Institute of Applied Mathematics, Putian University, Putian 351100, Fujian, China
[b] The Institute of Math. Sci., The Chinese University of Hong Kong, Shatin, N.T., Hong Kong

ARTICLE INFO

Article history:
Received 31 January 2015
Received in revised form 4 June 2015
Accepted 9 July 2015
Available online 3 August 2015

Keywords:
Removable singularity
Convection–diffusion
Source-type solution
Nonexistence
Critical exponent
Bernstein's estimate

ABSTRACT

In this paper we study the following Cauchy problem:

$$u_t = u_{xx} + (u^n)_x, \quad (x,t) \in \mathbb{R} \times (0,\infty),$$
$$u(x,0) = \delta(x), \quad x \in \mathbb{R},$$

where $\delta(x)$ is a Dirac measure and $n \geq 0$. Its solution is called source-type solution. Such singular solution plays an important role in the development of theory of nonlinear parabolic equations. However, there seems not to be perfect answer to the research. Here we focus on whether there is a critical exponent n_0 such that when $n < n_0$ there exists unique source-type solution, while $n \geq n_0$ there is no source-type solution, and on what the singular expansions of source-type solutions at origin are. From a physical point of view, there are phenomena of the interactive effect between the diffusion and convection in a heat process, which is re-confirmed and described through mathematical analysis and numerical simulation. In addition, thanks to the entropy inequality, we get new proof of uniqueness and are able to extend our approaches to nonlinear parabolic-hyperbolic equations with Radon measure as initial datum.

1. Introduction

In this paper we study the following equation

$$u_t = u_{xx} + (u^n)_x, \quad (x,t) \in S = \mathbb{R} \times (0,\infty) \tag{1.1}$$

with initial datum

$$u(x,0) = \delta(x), \quad x \in \mathbb{R}, \tag{1.2}$$

where $\delta(x)$ denotes the Dirac measure in $\mathbb{R}, n \geq 0$ is a constant.

* Corresponding author at: Institute of Applied Mathematics, Putian University, Putian 351100, Fujian, China.
E-mail address: gflu@sina.com (G. Lu).

http://dx.doi.org/10.1016/j.nonrwa.2015.07.001

Eq. (1.1) arises in many disciplines of science. For instance, it describes a heat flow in certain material with temperature dependent on conductivity, dopant diffusion in semi-conductors, or the movement of a thin liquid under gravity, to name a few. It is well known that such a nonnegative solution of the problem (1.1) (1.2) is called source-type solution of Eq. (1.1) and it is a singular solution. Source-type solutions play an important role in the development of theory of nonlinear parabolic equations. During the past decades, Kamin [1], Brezis and Friedman [2], Zhao [3] and Brezis [4] have studied the solvability of source-type solutions to the diffusion equation with absorptions, and Liu and Pierre [5] have considered source-type solutions of the conservation laws respectively. However, equations with nonlinear convection is very different from equations with absorption or in conservation law. In this case, equations are of a mixed parabolic-hyperbolic type. they possess a nonlinearity and hyperbolic feature due to the nonlinear convection. They have many interesting properties as described in [6,7]. The solvability of source-type solution of this type of equations depends not only on diffusion but also on convection. Meanwhile, the methods presented in early papers cannot be generalized to deal with those situations.

Due to the complex mathematical nature of the aforementioned singular solutions, there seem to be no previous works on the nonexistence and the short time asymptotic behavior of the solutions. Although Escobedo, Vazquez and Zuazua [6] have studied the existence of source-type solutions for the heat equation with convection, in our humble opinion they relied incorrectly on an unproven result to obtain their existence results, i.e., there is unique source-type solution of Eq. (1.1) when $n \geq 1$. In addition, in our previous work [8] we did not provide a complete description for the singular behavior of source-type solutions for very short time. Hence there is need for further investigation on these important problems. In this paper, we develop a number of analytical techniques such as Moser's iteration, an ODE method and scaling technique and use these techniques to obtain several accurate estimates for the approximating solutions and also create some special functions as subsolution and supersolution of Eq. (1.1) for comparison. The development of these techniques are based on our earlier works [8–11] and the methods developed in Chen and Zhang et al. [12,13]. As a result, we establish a nonexistence result and are able to describe the behavior of singularity expansion at the origin for very short time when such a singular solution exists (see Theorem A and Remark 1.1). Specifically we show the following:

If n is small, i.e. $0 < n < 2$, the effect of convection is negligible as compared to diffusion and there exists unique source-type solution to Eq. (1.1), and for very short time its singularity at origin expands with the same behavior as the fundamental solution of the heat equation;
If $n = 2$, Eq. (1.1) is the so-called Burger's equation and in this case an explicit self-similar source-type solution exists;
If n is larger, i.e. $2 < n < 3$, the convection is stronger than diffusion. In this case, there exists unique source-type solution to Eq. (1.1) and for very short time its singularity at origin expands with the same behavior as the nonnegative fundamental entropy solution in the conservation law;
If n is large enough, i.e. $n \geq 3$, the effect of convection dominate over diffusion. This leads to the fact that the diffusion and convection are not in coordination in the process, which makes impossible for the mass to concentrate at the origin in a very short time. Hence the singularity is removable and no source-type solutions of Eq. (1.1) exist.

As mentioned earlier, this paper emphasizes the interactive effects between the diffusion and the convection in the physical process and improves the results in Escobedo et al. [6] and in Lu et al. [8]. Further, Escobedo et al. [6] does not discuss the existence of source-type solution for $0 < n < 1$ and the nonexistence of such singular solution for $n \geq 3$. In view of no optimal estimates, they were unable to show the existence result in the cases $0 < n < 1$ and the nonexistence results in the case $n \geq 3$. In particular, they introduced the Cauchy problem which is equivalent to equation $v_t = v_{xx} + (v_x)^n$ with Heaviside's function as initial datum,