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The Cauchy problem for multiphase first-contact miscible models with viscous fingering

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ABSTRACT

In this paper, the Cauchy problem for multiphase first-contact miscible models with viscous fingering, is studied and a global weak solution is obtained by using a new technique from the Div–Curl lemma in the compensated compactness theorem. This work extends the previous works, (Juanes and Blunt, 2006; Barkve, 1989), which provided the analytical solutions and the entropy solutions respectively, of the Riemann problem, and (Lu, 2013), which provided the global solution of the Cauchy problem for the Keyfitz–Kranzer system.

1. Introduction

In this paper, we study the Cauchy problem for multiphase first-contact miscible models with viscous fingering

$$\begin{cases} S_t + f(S,T)_x = 0, \\ (ST - T)_t + (Tf(S,T) - T)_x = 0, \end{cases} \tag{1.1}$$

with bounded measurable initial data

$$(S(x,0), T(x,0)) = (S_0(x), T_0(x)), \quad 0 \le S_0(x) \le 1, \tag{1.2}$$

where S is the water saturation, $C = ST - T$ is the solvent concentration, and f is the water fractional flow function. System (1.1) is a special case of the following nonstrictly hyperbolic systems of conservation laws

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modelling polymer flooding in enhanced oil recovery

$$\begin{cases} S_t + f(S,T)_x = 0, \\ (ST + \beta(T))_t + (Tf(S,T) + \alpha(T))_x = 0. \end{cases} \tag{1.3}$$

When $\alpha(T) = aT, \beta(T) = bT$, where a, b are positive constants, system (1.3) represents a simple model for nonisothermal two-phase flow in a porous medium [1,2], and its Riemann problem was resolved and the entropy conditions were discussed in [1] under suitable conditions on f. When $\beta(T) = 0$ and $\alpha(T) = 0$, system (1.3) is the famous Keyfitz–Kranzer [3] or Aw–Rascle model [4], and the Riemann problem and the Cauchy problem of system (1.2) were studied in [3–18] and the references cited therein. When $\alpha(T) = 0$ and $\beta(T) \neq 0$, but $\beta'(T) > 0$, system (1.3) arises in connection with enhanced oil recovery, and its Riemann problem was resolved in [19]. For general $\alpha(T) \neq 0$ and $\beta(T) \neq 0$, but $\beta'(T) > 0$, the Cauchy problem was studied in the recent paper [20].

When $\beta'(T) < 0$, system (1.3) is of interest and difficulty in mathematics because the flux functions are singular. For instance, the functions

$$(f(S,T), Tf(S,T) - T) = \left(f\left(S, \frac{C}{S-1}\right), \frac{C}{S-1}\left(f\left(S, \frac{C}{S-1}\right) - 1\right)\right)$$

in system (1.1) are singular near the line $S = 1$.

In [21], the authors studied the analytical solutions of the Riemann problem for system (1.1).

As far as we know, there is no any existence result about the Cauchy problem of system (1.1) or system (1.3) when $\beta'(T) < 0$.

In this paper, we obtain the following results.

Theorem 1. *Let the initial data $(S_0(x), T_0(x))$ be bounded, $0 \leq S_0(x) \leq 1$, the total variation of the second variant $T_0(x)$ be bounded, the functions $f(S,T), \alpha(T), \beta(T)$ be suitable smooth and satisfy $f(0,T) = f(1,T) = 0$, meas $\{S : f_{SS}(S,T) = 0\} = 0$ for any fixed T, $\beta'(T) \leq -1$.*

(1) *If meas $\{T : \beta''(T) = 0\} = 0$ or $\beta'(T) = bT, b < -1$, then the Cauchy problem (1.3)–(1.2) has a global bounded entropy solution $(S(x,t), T(x,t))$ and $T_x(\cdot, t)$ is bounded in $L^1(R)$, namely, $(S(x,t), T(x,t))$ satisfies system (1.3) and the inequality $\eta(S,T)_t + q(S,T)_x \leq 0$, in the sense of distributions, for any smooth, convex, entropy $\eta(S,C)$ and the corresponding entropy flux $q(S,C)$.*
(2) *If $\beta'(T) = -T$ and $\alpha(T) = -T$, then the Cauchy problem (1.1)–(1.2) has a global bounded **weak** entropy solution $(S(x,t), T(x,t))$, namely, $(S(x,t), T(x,t))$ satisfies system (1.3) and the inequality $\eta(S,T)_t + q(S,T)_x \leq 0$, in the sense of distributions, for any smooth, convex, **weak** entropy $\eta(S,C)$ and the corresponding **weak** entropy flux $q(S,C)$.*

Definition 1. A **weak** entropy $\eta(S,C)$ of system (1.3) means $\eta(1,C) = c_1$, and a **weak** entropy flux $q(S,C)$ means $q(1,C) = c_2$, where c_1, c_2 are constants.

2. Proof of Theorem 1

To prove Theorem 1, we consider the Cauchy problem for the related parabolic system

$$\begin{cases} S_t + f(S,T)_x = \varepsilon S_{xx} \\ (ST + \beta(T) - \delta T)_t + (Tf(S,T) + \alpha(T))_x = \varepsilon(ST + \beta(T) - \delta T)_{xx}, \end{cases} \tag{2.1}$$

with initial data

$$(S^\varepsilon(x,0), T^\varepsilon(x,0)) = (S_0^\varepsilon(x), T_0^\varepsilon(x)), \tag{2.2}$$