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On the blow-up criterion of periodic solutions for micropolar equations in homogeneous Sobolev spaces

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ABSTRACT

We prove lower estimates for space periodic solutions $(\mathbf{u},\mathbf{w})(t)$ of the micropolar equations in their maximal interval $[0,T^*)$ provided that $T^*<\infty$. For example, we show for $0<\delta<1$ that $\|(\mathbf{u},\mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}$ is at least of the order $(T^*-t)^{-(\delta s)/(1+2\delta)}$ for $s\geq 1/2+\delta$. Moreover, we prove the inequality $\|(\widehat{\mathbf{u}},\widehat{\mathbf{w}})(t)\|_{l^1(\mathbb{Z}^3)}\geq C(T^*-t)^{-1/2}$, which yields the blow-up rate $(T^*-t)^{-s/3}$ for $\|(\mathbf{u},\mathbf{w})(t)\|_{\dot{H}^s(\mathbb{T}^3)}$ for $s>3/2$.

1. Introduction

In this paper we consider space periodic solutions for the following *micropolar system* in three dimensions:

$$\begin{cases} \mathbf{u}_t+\mathbf{u}\cdot\nabla\mathbf{u}+\nabla p=(\mu+\chi)\Delta\mathbf{u}+\chi\,\nabla\times\mathbf{w}, \\ \mathbf{w}_t+\mathbf{u}\cdot\nabla\mathbf{w}=\gamma\,\Delta\mathbf{w}+\kappa\,\nabla\mathrm{div}\,\mathbf{w}+\chi\,\nabla\times\mathbf{u}-2\chi\,\mathbf{w}, \\ \mathrm{div}\ \mathbf{u}=0, \\ \mathbf{u}(\cdot,0)=\mathbf{u}_0(\cdot), \qquad \mathbf{w}(\cdot,0)=\mathbf{w}_0(\cdot), \end{cases} \tag{1}$$

where $\mathbf{u}(x,t)=(u_1(x,t),u_2(x,t),u_3(x,t))\in\mathbb{R}^3$ denotes the velocity field, $\mathbf{w}(x,t)=(w_1(x,t),w_2(x,t),w_3(x,t))\in\mathbb{R}^3$ describes the micro-rotational velocity, and $p(x,t)\in\mathbb{R}$ the hydrostatic pressure. The spatial domain is the three-dimensional torus $\mathbb{T}^3=(\mathbb{R}\,\mathrm{mod}\,2\pi)^3$. With $x\in\mathbb{T}^3$ we denote the space variable and $0\leq t<T^*$ denotes the time variable. Here $[0,T^*)$ is the maximal interval of existence of the strong solution of (1) and we will always assume that T^* is finite. Our aim is to prove blow-up rates for various norms of the vector function $(\mathbf{u},\mathbf{w})(t)$ and its Fourier coefficients as time approaches the blow-up time T^*.

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The positive constants μ, χ, κ, and γ are associated with the specific properties of the fluid. More precisely, μ is the kinematic viscosity, χ is the vortex viscosity, κ and γ are spin viscosities. The initial data for the velocity field, given by $\mathbf{u}_0$ in (1), is divergence-free, i.e., div $\mathbf{u}_0 = 0$. To make the pressure unique, we impose the condition

$$\int_{\mathbb{T}^3} p(x,t)\,dx = 0, \quad \forall 0 \le t < T^*.$$

We also assume, without loss of generality, that

$$\int_{\mathbb{T}^3} (\mathbf{u}_0, \mathbf{w}_0)(x)\,dx = 0. \tag{2}$$

There are many works in the literature that prove the existence and uniqueness of solutions for problems related to the micropolar system (1) as, for example, [1–7]. In particular, G.P. Galdi and S. Rionero [4] considered the existence of weak solutions of the following initial boundary-value problem for the micropolar system

$$\begin{cases} \mathbf{u}_t + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = (\mu+\chi)\Delta\mathbf{u} + \chi\,\nabla\times\mathbf{w}, \quad x \in \Omega,\ t \in [0,T], \\ \mathbf{w}_t + \mathbf{u}\cdot\nabla\mathbf{w} = \gamma\,\Delta\mathbf{w} + \kappa\,\nabla\mathrm{div}\,\mathbf{w} + \chi\,\nabla\times\mathbf{u} - 2\chi\,\mathbf{w}, \quad x \in \Omega,\ t \in [0,T], \\ \mathrm{div}\ \mathbf{u} = 0, \quad x \in \Omega,\ t \in [0,T], \\ \mathbf{u}(x,0) = \mathbf{u}_0(x), \qquad \mathbf{w}(x,0) = \mathbf{w}_0(x), \quad x \in \Omega, \\ (\mathbf{u}(x,t), \mathbf{w}(x,t)) = 0, \quad (x,t) \in \partial\Omega \times [0,T]. \end{cases} \tag{3}$$

Here $\Omega \subset \mathbb{R}^3$ is a bounded domain with a sufficiently smooth boundary $\partial\Omega$. Also, for the system (3), J.L. Boldrini, M. Durán and M.A. Rojas-Medar [1] proved, by using the Galerkin method, the existence and uniqueness (local in time) of strong solution in $L^q(\Omega)$, for $q > 3$; here a compact C^2-boundary $\partial\Omega$ was assumed.

Considering the micropolar system (1) with spatial variable given in the whole space $\mathbb{R}^3$, J. Yuan [7] proved the next result, whose proof can be adapted to the space periodic case. For the definition of the operator Δ_j used in the theorem, we refer to [7].

Theorem 1.1 (*See [7]*)**.**

1. *Local existence: Let $s_0 > 3/2$ and assume that $(\mathbf{u}_0, \mathbf{w}_0) \in H^{s_0}(\mathbb{R}^3)$ with $\mathrm{div}\,\mathbf{u}_0 = 0$. Then there exists a positive $T^* = T^*(\|(\mathbf{u}_0, \mathbf{w}_0)\|_{H^{s_0}(\mathbb{R}^3)})$, with $0 < T^* \le \infty$ so that a unique strong solution $(\mathbf{u}, \mathbf{w})(t) \in C^0([0,T^*); H^{s_0}(\mathbb{R}^3)) \cap C^1((0,T^*); H^{s_0}(\mathbb{R}^3)) \cap C^0((0,T^*); H^{s_0+2}(\mathbb{R}^3))$ for the system (1) exists;*
2. *Blow-up criterion: Assume that $s_0 > 3/2$ and let $(\mathbf{u}, \mathbf{w})(t) \in C^0([0,T^*); H^{s_0}(\mathbb{R}^3)) \cap C^1((0,T^*); H^{s_0}(\mathbb{R}^3)) \cap C^0((0,T^*); H^{s_0+2}(\mathbb{R}^3))$ denote the smooth solution for the system (1) in $0 \le t < T^*$. There is an absolute constant $M > 0$ with the following property: If*

$$\lim_{\epsilon\to 0}\sup_{j\in\mathbb{Z}} \int_{T^*-\epsilon}^{T^*} \|\Delta_j(\nabla\times\mathbf{u})(t)\|_\infty\,dt := \delta < M,$$

then $\delta = 0$ and the solution $(\mathbf{u}, \mathbf{w})(t)$ can be extended past time $t = T^*$. If

$$\lim_{\epsilon\to 0}\sup_{j\in\mathbb{Z}} \int_{T^*-\epsilon}^{T^*} \|\Delta_j(\nabla\times\mathbf{u})(t)\|_\infty\,dt \ge M,$$

then the solution $(\mathbf{u}, \mathbf{w})(t)$ blows-up at $t = T^*$.

It is important to point out that if $T^* < \infty$ is the blow-up instant for the solution $(\mathbf{u}, \mathbf{w})(t)$ given by Theorem 1.1, then one obtains $(\mathbf{u}, \mathbf{w}) \in C^\infty(\mathbb{T}^3 \times (0,T^*))$, with $(\mathbf{u}, \mathbf{w})(t) \in C^0((0,T^*); H^s(\mathbb{T}^3))$ for all