Nonlinear Analysis: Real World Applications 27 (2016) 146–157

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

On a nonlocal degenerate parabolic problem

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ARTICLE INFO

Article history:
Received 25 July 2014
Received in revised form 17 July 2015
Accepted 28 July 2015
Available online 14 August 2015

Keywords:
Nonlocal
Degenerate
Parabolic
PDE

ABSTRACT

Conditions for the existence and uniqueness of weak solutions for a class of nonlinear nonlocal degenerate parabolic equations are established. The asymptotic behaviour of the solutions as time tends to infinity are also studied. In particular, the finite time extinction and polynomial decay properties are proved.

1. Introduction

In this paper, we study parabolic problems with nonlocal nonlinearity of the following type:

$$\begin{cases} u_t - \left(\displaystyle\int_\Omega u^2(x,t)dx\right)^\gamma \Delta u = f\,(x,t)\,, \quad (x,t) \in \Omega\times]0,T] \\ u\,(x,t) = 0, \quad (x,t) \in \partial\Omega\times]0,T] \\ u(x,0) = u_0(x), \quad x \in \Omega \end{cases} \tag{1}$$

where Ω is a bounded open domain in $\mathbb{R}^d, d \geq 1, \gamma$ is a real constant, f and u_0 are continuous integrable functions.

This type of problem was studied initially by Chipot and Lovat in [1], where they proposed the equation

$$u_t - a\left(\int_\Omega u\ dx\right) \Delta u = f \tag{2}$$

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http://dx.doi.org/10.1016/j.nonrwa.2015.07.015
1468-1218/

for modelling the density of a population, for example, of bacteria, subject to spreading. This equation also appears in the study of heat propagation or in epidemic theory. In this paper the authors prove the existence and uniqueness of a weak solution to this equation.

In [1], the authors studied the problem

$$\begin{cases} u_t - a(l(u))\Delta u = f(x,t) & \text{in } \Omega \times (0,T) \\ u(x,t) = 0 & \text{on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x) & \text{in } \Omega \end{cases} \tag{3}$$

where Ω is a bounded open subset in $\mathbb{R}^d, d \geq 1$, with smooth boundary $\partial\Omega, T$ is some arbitrary time and a is some function from $\mathbb{R}$ into $(0,+\infty)$. Both a and f are continuous functions and $l : L_2(\Omega) \to \mathbb{R}$ is a continuous linear form. The existence, uniqueness and asymptotic behaviour of weak and strong solutions of parabolic equations and systems with nonlocal diffusion terms have been widely studied in the last two decades.

In 2000, Ackleh and Ke [2] studied the problem

$$\begin{cases} u_t = \dfrac{1}{a\left(\int_\Omega u\ dx\right)}\Delta u + f(u) & \text{in } \Omega\times]0,T] \\ u(x,t) = 0 & \text{on } \partial\Omega\times]0,T] \\ u(x,0) = u_0(x) & \text{in } \overline{\Omega} \end{cases},$$

with $a(\xi) > 0$ for all $\xi \neq 0, a(0) \geq 0$ and f Lipschitz-continuous satisfying $f(0) = 0$. They proved the existence and uniqueness of a solution to this problem and gave conditions on u_0 for the extinction in finite time and for the persistence of solutions. The asymptotic behaviour of the solutions as time tends to infinity was studied by Zheng and Chipot [3] for nonlinear parabolic equations with two classes of nonlocal terms, in a cylindrical domain. Recently, Duque et al. [4] considered a nonlinear coupled system of reaction–diffusion on a bounded domain with a more general nonlocal diffusion term working on two linear forms l_1 and l_2:

$$\begin{cases} u_t - a_1(l_1(u), l_2(v))\Delta u + \lambda_1|u|^{p-2}u = f_1(x,t) & \text{in } \Omega\times]0,T] \\ v_t - a_2(l_1(u), l_2(v))\Delta v + \lambda_2|v|^{p-2}v = f_2(x,t) & \text{in } \Omega\times]0,T] \end{cases}. \tag{4}$$

In this case, u and v could describe the densities of two populations that interact through the functions a_1 and a_2. The death in species u is assumed to be proportional to $|u|^{p-2}u$ by the factor $\lambda_1 \geq 0$ and in species v to be proportional to $|v|^{p-2}v$ by the factor $\lambda_2 \geq 0$ with $p \geq 1$. The supply of being by external sources is denoted by f_1 and f_2. The authors proved the existence and uniqueness of weak and strong global in time solutions and imposed conditions, on the data, for these solutions to have the waiting time and stable localization properties. Moreover, important results on polynomial and exponential decay and vanishing of the solutions in finite time were also presented.

Robalo et al. [5] proved the existence and uniqueness of weak and strong global in time solutions and gave conditions, on the data, for these solutions to have the exponential decay property for a nonlocal problem with moving boundaries.

The numerical analysis and simulation of such problems were less studied (see, for example, [6–10] and their references).

In this work, we analyse a different diffusion term, dependent on the L_2-norm of the solution. In most of the previous papers, it is assumed that the diffusion term is bounded, with $0 < m \leq a(s) \leq M < \infty, s \in \mathbb{R}$, so the problem is always nondegenerate. Here, we study a case were the diffusion term could be zero or infinity. This work is concerned with the proof of the existence, uniqueness and asymptotic behaviour of the weak solutions. To the best of our knowledge, for nonlocal reaction–diffusion equations with this type of diffusion term, these results have not yet been established.

The paper is organized as follows. In Section 2, we formulate the problem and the hypotheses on the data. In Section 3, we define an auxiliary problem and prove the existence of weak solutions for the initial