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Existence and stability of solutions to the compressible Euler equations with an outer force

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ABSTRACT

We study the compressible Euler equation with an outer force. The global existence theorem has been proved in many papers, provided that the outer force is bounded. However, the stability of their solutions has not yet been obtained until now. Our goal in this paper is to prove the existence of a global solution without such an assumption as boundedness. Moreover, we deduce a uniformly bounded estimate with respect to the time. This yields the stability of the solution.

When we prove the global existence, the most difficult point is to obtain the bounded estimate for approximate solutions. To overcome this, we employ an invariant region, which depends on both space and time variables. To use the invariant region, we introduce a modified difference scheme. To prove their convergence, we apply the compensated compactness framework.

1. Introduction

The present paper is concerned with the compressible Euler equation with an outer force.

$$\begin{cases} \rho_t + m_x = 0, \\ m_t + \left(\dfrac{m^2}{\rho} + p(\rho) \right)_x = F(x,t)\rho, \quad x \in \mathbf{R}, \end{cases} \tag{1.1}$$

where ρ, m and p are the density, the momentum and the pressure of the gas, respectively. If $\rho > 0$, $v = m/\rho$ represents the velocity of the gas. For a barotropic gas, $p(\rho) = \rho^\gamma/\gamma$, where $\gamma \in (1, 5/3]$ is the adiabatic exponent for usual gases. The given function $F(x,t) \in C^1(\mathbf{R} \times \mathbf{R}_+)$ represents the outer force.

We consider the Cauchy problem (1.1) with the initial data

$$(\rho, m)|_{t=0} = (\rho_0(x), m_0(x)). \tag{1.2}$$

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The above problem (1.1)–(1.2) can be written in the following form

$$\begin{cases} u_t + f(u)_x = g(x,u), \quad x \in \mathbf{R}, \\ u|_{t=0} = u_0(x), \end{cases} \tag{1.3}$$

by using $u = {}^t(\rho, m)$, $f(u) = {}^t\left(m, \frac{m^2}{\rho} + p(\rho)\right)$ and $g(x,u) = {}^t(0, F(x,t)\rho)$.

In the present paper, we consider the compressible Euler equation. Let us survey the related mathematical results.

Concerning the one-dimensional Cauchy problem, DiPerna [1] proved the global existence by the vanishing viscosity method and a compensated compactness argument. The method of compensated compactness was introduced by Murat [2] and Tartar [3,4]. DiPerna first applied the method to systems for the special case where $\gamma = 1 + 2/n$ and n is an odd integer. Subsequently, Ding, Chen and Luo [5] and Chen [6] extended his analysis to any γ in $(1, 5/3]$. In [7], Ding, Chen and Luo treated isentropic gas dynamics with a source term by using the fractional step procedure.

To state our main theorem, we define the Riemann invariants w, z, which play important roles in this paper, as

Definition 1.

$$w := \frac{m}{\rho} + \frac{\rho^\theta}{\theta} = v + \frac{\rho^\theta}{\theta}, \qquad z := \frac{m}{\rho} - \frac{\rho^\theta}{\theta} = v - \frac{\rho^\theta}{\theta} \qquad \left(\theta = \frac{\gamma - 1}{2}\right).$$

These Riemann invariants satisfy the following.

Remark 1.1.

$$|w| \geqq |z|, \quad w \geqq 0, \text{ when } v \geqq 0. \qquad |w| \leqq |z|, \quad z \leqq 0, \text{ when } v \leqq 0. \tag{1.4}$$

$$v = \frac{w+z}{2}, \qquad \rho = \left(\frac{\theta(w-z)}{2}\right)^{1/\theta}, \qquad m = \rho v. \tag{1.5}$$

From the above, the lower bound of z and the upper bound of w yield the bound of ρ and $|v|$.

Moreover, we define the entropy weak solution.

Definition 2. A measurable function $u(x,t)$ is called a global *entropy weak solution* of the Cauchy problems (1.3) if

$$\int_{-\infty}^{\infty} \int_0^{\infty} u\phi_t + f(u)\phi_x + g(x,u)\phi dx dt + \int_{-\infty}^{\infty} u_0(x)\phi(x,0)dx = 0$$

holds for any test function $\phi \in C_0^1(\mathbf{R} \times \mathbf{R}_+)$ and

$$\int_{-\infty}^{\infty} \int_0^{\infty} \eta(u)\psi_t + q(u)\psi_x + \nabla\eta(u)g(x,u)\psi dx dt + \int_{-\infty}^{\infty} \eta(u_0(x))\psi(x,0)dx \geqq 0$$

holds for any non-negative test function $\psi \in C_0^1(\mathbf{R} \times \mathbf{R}_+)$, where (η, q) is a pair of convex entropy–entropy flux of (1.1).

We assume that there exist functions $X \in C^1(\mathbf{R}) \cap L^1(\mathbf{R})$ and $T \in C^1(\mathbf{R}_+)$ such that

$$|F(x,t) - T(t)| \leqq X(x) \quad t \in \mathbf{R}_+, \tag{1.6}$$

and $I \in L^\infty(\mathbf{R}_+)$, where $I(t) = \int_0^t T(s)ds$.