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Existence of positive solutions for the cantilever beam equations with fully nonlinear terms☆

Yongxiang Li*

Department of Mathematics, Northwest Normal University, Lanzhou 730070, People's Republic of China

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ABSTRACT

In this paper we discuss the existence of positive solutions of the fully fourth-order boundary value problem

$$\begin{cases} u^{(4)} = f(t, u, u', u'', u'''), & t \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases}$$

which models a statically elastic beam fixed at the left and freed at the right, and it is called cantilever beam in mechanics, where $f : [0, 1] \times {\mathbb{R}_+}^3 \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous. Some inequality conditions on f guaranteeing the existence of positive solutions are presented. Our conditions allow that $f(t, x_0, x_1, x_2, x_3)$ is superlinear or sublinear growth on x_0, x_1, x_2, x_3. For the superlinear case, a Nagumo-type condition is presented to restrict the growth of f on x_2 and x_3. Our discussion is based on the fixed point index theory in cones.

1. Introduction

The deformations of an elastic beam in equilibrium state, whose one end-point is fixed and the other freed, can be described by fourth-order ordinary differential equation boundary value problem (BVP)

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t), u''(t), u'''(t)), & t \in [0, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.1}$$

where $f : [0, 1] \times {\mathbb{R}_+}^3 \times \mathbb{R}_- \to \mathbb{R}_+$ is continuous. In mechanics, the problem is called cantilever beam equation, and in the equation, the physical meaning of the derivatives of the deformation function $u(t)$ is as

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* Tel.: +86 0931 7971111.
E-mail address: liyxnwnu@163.com.

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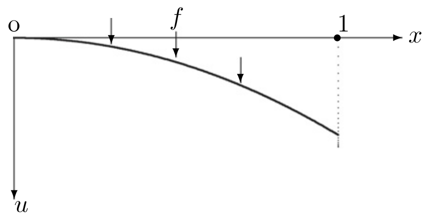

Fig. 1. Cantilever beam.

follows: $u^{(4)}$ is the load density stiffness, u''' is the shear force stiffness, u'' is the bending moment stiffness, and u' is the slope [1–4]. In some practice, only its positive solution is significant, see Fig. 1. In this paper, we discuss the existence of positive solutions of BVP (1.1).

For the special case of BVP (1.1) that f does not contain any derivative terms, namely the simply fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t,\, u(t)), \quad t \in [0,\, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.2}$$

and f only contains first-order derivative term u', namely the fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t,\, u(t),\, u'(t)), \quad t \in [0,\, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0, \end{cases} \tag{1.3}$$

the existence of positive solutions has been discussed by some authors, see [5–9]. In Refs. [5–7], the BVP (1.2) appears as a special case of the $(p,\, n-p)$ focal boundary value problems for $p = 2$ and $n = 4$. In these works, the positivity of $G(t,\, s)$ and $\frac{\partial}{\partial t}\, G(t,\, s)$ plays an important role, where $G(t,\, s)$ is the Green function of the corresponding linear boundary value problem

$$\begin{cases} u^{(4)}(t) = 0, \quad t \in [0,\, 1], \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases} \tag{1.4}$$

The positivity guarantees that the BVP (1.2) or BVP (1.3) can be converted to a fixed point problem of a cone mapping in $C(I)$ or $C^1(I)$, where $I = [0,\, 1]$. Hence, these authors can apply the fixed point theorems of cone mapping to obtain the existence of positive solutions for BVP (1.2) or BVP (1.3). But their argument methods are not applicable to BVP (1.1), since these methods cannot deal with the derivative terms u'' and u'''.

For the cantilever beam equation with a nonlinear boundary condition of third order derivative

$$\begin{cases} u^{(4)}(t) = f(t,\, u(t),\, u'(t)), \quad t \in [0,\, 1], \\ u(0) = u'(0) = 0, \qquad u''(1) = 0, \qquad u'''(1) = g(u(1)), \end{cases} \tag{1.5}$$

the existence of solution has also been discussed by some authors, see [10–13]. The boundary condition in (1.5) means that the left end of the beam is fixed and the right end of the beam is attached to an elastic bearing device, see [10].

For the elastic beam equations with other boundary conditions, the existence of positive solutions has been studied by many authors, see [14–22]. In Refs. [14–17], the simple beam equation

$$\begin{cases} u^{(4)}(t) = f(t,\, u(t)), \quad t \in [0,\, 1], \\ u(0) = u(1) = u''(0) = u''(1) = 0, \end{cases} \tag{1.6}$$

has been studied, in which the boundary conditions mean that two ends of the beam are simply supported. One well-known result is if $f(t,\, u)$ is superlinear or sublinear growth on u, then the BVP (1.6) has a positive