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Fast decay of solutions for wave equations with localized dissipation on noncompact Riemannian manifolds

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ABSTRACT

In this paper, uniform energy and L^2 decay for solutions of linear wave equations with an energy term and localized dissipation on certain noncompact Riemannian manifolds are considered. We prove that the total energy of the solutions decay like $O(1/t^2)$ as t goes to infinity under some assumptions on the curvature of the manifolds and initial data. It is shown that the decay depends not only on the initial data but also on the curvature properties of the manifolds. As an application, we obtain the decay rate for the solutions of the wave equation with variable coefficients on an exterior domain of R^n.

1. Introduction and main results

Let (M, g) be a n-dimensional noncompact Riemannian manifold for $n \geq 3$. In this paper, we consider the following wave equation on this manifold

$$\begin{cases} u_{tt}(t,x) - \Delta_M u(t,x) - \langle DP(x), Du\rangle_g + a(x)u_t(t,x) = 0, & (t,x) \in (0,+\infty) \times M, \\ u(0,x) = u_0(x), \qquad u_t(0,x) = u_1(x), & x \in M, \\ u|_{\partial M} = 0, \end{cases} \tag{1.1}$$

where Δ_M is the associated Laplace–Beltrami operator and D is the Levi Civita connection on manifold (M, g), respectively. $a(x) \in L^\infty(M)$ is a nonnegative function on (M, g). $P(x) \in W^{2,\infty}(M)$ is the coefficient of the energy term.

We define the total energy of the problem (1.1) as

$$E(t) = \frac{1}{2}\|u_t(t,\cdot)\|^2 + \frac{1}{2}\|Du\|^2, \tag{1.2}$$

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where $\|\cdot\|$ denotes the usual $L^2(M)$-norm. Furthermore, we set

$$(f,h)_{L^2M} = \int_M f(x)h(x)dg.$$

We are interested in the decay rate of $E(t)$. We find that it depends on the curvature properties of the manifolds.

In the Euclidean case $(M,g) = (\Omega,\delta)$, where Ω is an exterior domain of $\mathbb{R}^n$ and δ is the Euclidean metric, the decay estimate of the solutions to problem (1.1) has been widely discussed, see [1–8] and many others. In the case when $a(x) \equiv constant > 0$ in all of Ω, the decay estimates in the framework of weighted initial data have been derived in [2,9]:

$$E(t) \le C(1+t)^{-2}, \qquad \|u(t,\cdot)\|^2 \le C(1+t)^{-1}. \tag{1.3}$$

Next for the case of the localized dissipation Ryo [4] has obtained the total energy decay estimate like (1.3) with certain geometrical condition on the boundary. He assumed that the dissipation $a(x)u_t$ is effective near infinity. As for Cauchy problem with initial data with compact support, [10] established $\|u(t)\|_{L^2} = O(t^{-n/4})$ when $a(x) \equiv constant > 0$. [11] treated the x-dependent potential under the assumption $a(x) \ge \frac{a_0}{(1+|x|)^\beta}$ and obtained the energy decay of $\|u(t)\|_{L^2} = o(t^{\frac{2\beta-n}{2(2-\beta)}+\varepsilon})$. For a detailed review of the pertinent literature, see [11,12].

However, as for the most general noncompact manifolds (M,g), it seems unknown whether the total energy decay estimate like (1.3) can be derived or not, especially when some lower order terms are involved in the equations. The purpose of this paper is to derive the fast decay rate like (1.3) with the weighted initial data by use of the "localized" dissipation term u_t. We find that the domain where the dissipation located is essentially determined by the radical curvature of the Riemannian manifold (M,g).

Before introducing our results we shall state some notations and assumptions.

Let $p \in M$ be a fixed point and let $\rho = \rho(x)$ be the distance function from $x \in M$ to p in the metric g. Denote $exp_p : M_p \to M$ the exponential map and $S \subset M_p$ the unit sphere of M_p. Therefore $\gamma(t) = \exp_p tv$ is a normal geodesic where $v \in S$. For any unit vector $X \in M_{\gamma(t)}$ which is orthogonal to the vector $\dot\gamma(t)$, $R(\dot\gamma(t), X, \dot\gamma(t), X)$ is the radial curvature along $\gamma(t)$, for all $t \ge 0$. Let the nonnegative continuous functions $k, K : [0,\infty) \to [0,\infty)$ be defined by

$$-k(s) = \min\{0, \text{ radial curvature at any } x \in M \text{ where } \rho(x) = s\}, \tag{1.4}$$

$$K(s) = \max\{0, \text{ radial curvature at any } x \in M \text{ where } \rho(x) = s\}. \tag{1.5}$$

We refer the readers to [13–16] for further relationships.

For our purposes, we make the following assumptions:

(A1)

$$1 - \int_0^\infty sK(s)ds > \frac{1}{n-1} \quad \text{and} \quad \int_0^\infty sk(s)ds < \infty. \tag{1.6}$$

This assumption is about the radial curvature.

(A2) Let Ω_0 be a bounded open subset with $p \in \bar\Omega_0 \subset M$. For the nonnegative function $a(x) \ge 0$, there exists $\varepsilon_0 > 0$ such that

$$a(x) \ge \varepsilon_0, \quad x \in M \setminus \Omega_0. \tag{1.7}$$

Thus the dissipation is so called a "localized" one.