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Finsler geometry for nonlinear path of fluids flow through inhomogeneous media

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ABSTRACT

Fluids flow followed by Darcy's law through inhomogeneous porous media is studied by the theory of Finsler geometry. According to Fermat's variational principle, the nonlinear paths of fluids flow called Darcy's flow are described by geodesics in a Finsler space. For inhomogeneous media, the direction dependence of Darcy's flow gives a Finsler metric called Kropina metric. Then, the influence of direction dependence on the Darcy's flow is shown by the differences between Riemannian geodesics and Finslerian geodesics. In this case, the deviation curvature tensor implies that the trajectory of Darcy's flow is Jacobi unstable for the deviation of geodesics. Moreover, similar to Darcy's flow, the seismic ray path in anisotropic media can be defined in Finsler space, and the metric of seismic ray path is given by the m th root metric. It is shown that the relationship between the variational problems of Darcy's flow and seismic ray path in Finsler space.

1. Introduction

The ray theory based on Fermat's variational principle is closely related to differential geometry [1,2]. In the simplest case, a shortest ray path between two different points is given by a straight line in Euclidean space. However, the real ray path is not a straight line because of physical or chemical property of media. Therefore, from a viewpoint of geometry, the ray theory is actually formulated in the non-Euclidean space.

For the non-Euclidean ray theory, the anisotropy of media such as crystals or rocks is connected with a geometric property of ray path. The anisotropy of media corresponds to the direction dependence of metric tensor. This means that the ray theory for anisotropic media is defined in a non-Riemannian space called Finsler space [1,3,4]. For example, the propagation of elastic wave or the seismic wave through anisotropic media has been discussed by the theory of Finsler geometry [5–9]. The ray path is regarded as the geodesics, and the elementary wavefront corresponds to the indicatrix in the Finsler space. In this case, the gradient of the wavefront function for the anisotropic media is perpendicular to the wavefront in the Hamiltonian sense [10]. Especially, the anisotropy of rocks is related to the parameter of mth root metric in Finsler space [11].

Similar to the seismic ray theory, the fluids flow or heat flow through inhomogeneous media is also based on Fermat's variational principle [12,13]. The motion of fluids flow is given by Darcy's law, which states that the velocity of fluid flow is proportional to the gradient of hydraulic head. In this case, since the hydraulic conductivity depends on the position by the inhomogeneity, the fluid streamline is not a straight line. This means that the fluids flow through inhomogeneous

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porous media is geometrically formulated in the non-Euclidean space. It has been mentioned that the Finslerian framework of Darcy's flow which depends on the direction caused by a change of area perpendicular to flow [13]. Therefore, Darcy's flow will be discussed based on the fundamental function and the geometric objects in the Finsler space.

In this study, the fluids flow through inhomogeneous porous media is formulated by the Finsler geometry. In Section 2, the Finsler metric of Darcy's flow is obtained. In Section 3, the geodesics as the path of Darcy's flow are investigated. Then, in Section 4, the stability of Darcy's flow is discussed based on curvature tensor. Moreover, in Section 5, the relationship between the Darcy's flow and other Finslerian ray theory, the seismic ray theory is shown. Finally, Section 6 is the conclusion. In this paper, we use Einstein's summation convention.

2. Finsler metric of Darcy's flow through inhomogeneous media

2.1. Brief review of Fermat's variational principle for Darcy's flow

In this study, we consider a two-dimensional path of fluids flow based on [13]. Let $(x^i) = (x^1, x^2)$ be the local coordinates of the cross-section in the Earth's interior. Here, x^1 is horizontally directed to the Earth's surface, and x^2 is vertically directed to the Earth's interior. From [13], the path of fluids flow through inhomogeneous porous media between two fixed points x_a and x_b is given by Fermat's variational principle which renders the total resistance of flow minimum:

$$\delta J = \delta \int_{x_a}^{x_b} p dl = \delta \int_{x_a}^{x_b} \frac{I^2 \rho}{A} dl = 0, \tag{1}$$

where $p = I^2 \rho / A$. A, l and I are the variable cross-sectional area of flux tube perpendicular to the flow, the length of flow and the flux of flow, respectively. ρ is the resistance or the reciprocal of conductivity κ given by Darcy's law:

$$v^i = \kappa(x^j) \frac{\partial u}{\partial x^i}, \tag{2}$$

where v^i and u are the rate of fluids' flow and the hydraulic head. According to [13], the fluids flow described by (1) and (2) is called Darcy's flow. In the following, from a general viewpoint, we formulate the general case that ρ is a function of both x^1 and x^2: $\rho(x^i) = \rho(x^1, x^2)$.

2.2. Geometric framework and Finsler function of Darcy's flow

Based on the above conditions, we derive a Finsler metric of Darcy's flow. Let M be a two-dimensional manifold corresponding to the cross-section in the Earth's interior. $(x^i) = (x^1, x^2)$ are the local coordinates on M. Moreover, let TM be a tangent bundle over M, and (x^i, y^i) are the local coordinates on TM, where $y^i = dx^i/dt$, and t is a time. Then, the variational problem (1) is rewritten by

$$\delta J = \delta \int_{t_a}^{t_b} \frac{I^2 \rho(x^i)}{A} \sqrt{(y^1)^2 + (y^2)^2} dt = 0. \tag{3}$$

Fermat's variational principle determines the shape of ray path of Darcy's flow. The incident angle θ between the ray of Darcy's flow and the horizontal x^1-axis is given by

$$y' \equiv \tan\theta = \frac{dx^2}{dx^1} = \frac{y^2}{y^1}, \tag{4}$$

$$\cos\theta = \frac{dx^1}{\sqrt{(dx^1)^2 + (dx^2)^2}}, \qquad \sin\theta = \frac{dx^2}{\sqrt{(dx^1)^2 + (dx^2)^2}}. \tag{5}$$

When the constant area of flux tube projected on x^2-axis is denoted by A_0, the variable area A perpendicular to flow is expressed by $A = A_0 \cos\theta$ [13]. In this case, from (5), the cross-sectional area A depends on the direction of ray:

$$A(y^i) = \frac{A_0 y^1}{\sqrt{(y^1)^2 + (y^2)^2}}. \tag{6}$$

The dependence of y^i in (6) implies that the Darcy's flow is formulated by the Finsler geometry [13]. From (3) and (6), the fundamental function of Darcy's flow is given by

$$L(x^i, y^i) = \frac{I^2 \rho(x^i)}{A_0} \left\{ \frac{(y^1)^2 + (y^2)^2}{y^1} \right\}. \tag{7}$$