Nonlinear Analysis: Real World Applications 25 (2015) 79–95

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

www.elsevier.com/locate/nonrwa

Chaotic dynamics in three dimensions: A topological proof for a triopoly game model

Marina Pireddu*

Department of Mathematics and Applications, University of Milano-Bicocca, Milano, Italy

ARTICLE INFO

Article history:
Received 15 May 2013
Accepted 10 March 2015
Available online 2 April 2015

Keywords:
Chaotic dynamics
Stretching along the paths
Triopoly games
Heterogeneous players

ABSTRACT

We rigorously prove the existence of chaotic dynamics for a triopoly game model. In the model considered, the three firms are heterogeneous and in fact each of them adopts a different decisional mechanism, i.e., linear approximation, best response and gradient mechanisms, respectively. The method we employ is the so-called "Stretching Along the Paths" (SAP) technique, based on the Poincaré–Miranda Theorem and on the properties of the cutting surfaces.

1. Introduction

In the economic literature, due to the complexity of the models considered, an analytical study of the associated dynamical features turns out often to be too difficult or simply impossible to perform. That is why many dynamical systems are studied mainly from a numerical viewpoint (see, for instance, [1–4]). Sometimes, however, even such kind of study turns out to be problematic, especially with high-dimensional systems, where several variables are involved [5].

In particular, as observed in Naimzada and Tramontana's working paper [6], this may be the reason for the relatively low number of works on triopoly games (see, for instance, [7–9]), where the context is given by an oligopoly composed by three firms. In such framework, a local analysis can generally be performed in the special case of homogeneous triopoly models, i.e., those in which the equations describing the dynamics are symmetric (see, for instance, [10–12]).

A more difficult task is that of studying heterogeneous triopolies, where the three firms considered behave according to different strategies. In fact, in the absence of complete information, both in regard to the shape of the demand function and with respect to the competitors' future output choices, in those models it is assumed that at each time period firms decide how much to produce in the following period according to

* Correspondence to: University of Milano-Bicocca, U5 Building, Via Cozzi 55, 20125 Milano, Italy. Tel.: +39 0264485767; fax: +39 0264485705.

E-mail address: marina.pireddu@unimib.it.

http://dx.doi.org/10.1016/j.nonrwa.2015.03.003
1468-1218/

different behavioral mechanisms. See [13–15] for some works on oligopolies with boundedly rational players, while the study of heterogeneous triopolies has been performed, for instance, in [16,17], as well as in the above mentioned paper by Naimzada and Tramontana [6]. In this latter work, in addition to the classical heterogeneity with interacting agents adopting gradient and best response mechanisms, it is assumed that one of the firms adopts a linear approximation mechanism, which means that the firm does not know the shape of the demand function and thus builds a conjectured demand function through the local knowledge of the true demand function. In regard to such model, those authors perform a stability analysis of the Nash equilibrium and show numerically that, according to the choice of the parameter values, it undergoes a flip bifurcation or a Neimark–Sacker bifurcation leading to chaos.

What we then aim to do in the present paper is complementing that analysis, by proving the existence of chaotic sets for the model in [6] only via topological arguments. This task will be performed using the "Stretching Along the Paths" (from now on, SAP) technique, already employed in [18] to rigorously prove the presence of chaos for some discrete-time one- and bi-dimensional economic models of the classes of overlapping generations and duopoly game models. Notice however that, to the best of our knowledge, this is the first three-dimensional discrete-time application of the SAP technique, called in this way because it concerns maps that expand the arcs along one direction and are instead compressive in the remaining directions. We stress that, differently from other methods for the search of fixed points and the detection of chaotic dynamics based on more sophisticated algebraic or geometric tools, such as the Conley index or the Lefschetz number (see, for instance, [19–21]), the SAP method relies on relatively elementary arguments and it is easy to apply in practical contexts, without the need of ad-hoc constructions. No differentiability conditions are required for the map describing the dynamical system under analysis and even continuity is needed only on particular subsets of its domain. Moreover, the SAP technique can be used to rigorously prove the presence of chaos also for continuous-time dynamical systems. In fact, in such framework it suffices to apply the results in Section 2, suitably modified, to the Poincaré map associated to the considered system[1] and thus one is led back to work with a discrete-time dynamical system. However, the geometry required to apply the SAP method turns out to be quite different in the two contexts: in the case of discrete-time dynamical systems we look for "topological horseshoes" (see, for instance, [24–26]), that is, a weaker version of the celebrated Smale horseshoe in [27], while in the case of continuous-time dynamical systems one has to consider the case of switching systems and the needed geometry is usually that of the so-called "Linked Twist Maps" (LTMs) (see [28–30]), as shown for the planar case in [22,23]. We also stress that the Poincaré map is a homeomorphism onto its image, while in the discrete-time framework the function describing the considered dynamical system need not be one-to-one, like in our example in Section 3. Hence, in the latter context, it is in general not be possible to apply the results for the Smale horseshoe, where one deals with homeomorphisms or diffeomorphisms. As regards three-dimensional continuous-time applications of the SAP method, those have recently been performed in [31], in a higher-dimensional counterpart of the LTMs framework, and in [32], where a system switching between different regimes is considered.

For the reader's convenience, we are going to recall in Section 2 what are the basic mathematical ingredients behind the SAP method, as well as the main conclusions it allows to draw about the chaotic features of the model under analysis. It will then be shown in Section 3 how it can be applied to the triopoly game model taken from [6]. Some further considerations and comments can be found in Section 4, which concludes the paper.

[1] We stress that, in order to apply the SAP method to continuous-time systems, as done in [22,23], it is not required to know the analytic formulation of the corresponding Poincaré map, but in general it suffices to know the geometry of the orbits in the phase-plane.