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Blow up of classical solutions to the isentropic compressible Navier–Stokes equations

Ning-An Lai*

Department of Mathematics, Lishui University, Lishui City 323000, China

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ABSTRACT

In this work we consider the isentropic compressible Navier–Stokes equations in three space dimensions. Blow up result will be established, assuming the gradient of the velocity satisfies some decay constraint and the initial total momentum does not vanish. We prove the main result by a contradiction argument, based on the conservation of the total mass and the total momentum.

1. Introduction

We study the Cauchy problem for the three-dimensional isentropic compressible Navier–Stokes equations:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \mu \Delta u + (\lambda + \mu) \nabla \nabla \cdot u, \\ p = A\rho^\gamma, \end{cases} \tag{1}$$

with initial data

$$\rho(0, x) = \rho_0(x), \qquad u(0, x) = u_0(x). \tag{2}$$

Here $\rho : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ denotes the density, $u : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}^3$ is the velocity field and $p : \mathbb{R}_+ \times \mathbb{R}^3 \to \mathbb{R}$ is the scalar pressure. The parameters μ, λ, A and γ are constants which satisfy

$$A > 0,$$

* Tel.: +86 18357880751.
E-mail address: hyayue@gmail.com.

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and

$$\gamma > 1, \qquad \mu > 0, \qquad \lambda + \frac{2\mu}{3} \geq 0, \tag{3}$$

where μ and λ denote the coefficient of viscosity and the second coefficient of viscosity.

Before giving our main result, we first make a brief survey for compressible Navier–Stokes system. As one of the most important systems in fluid dynamics, problem (1) has attracted much attention, including both the Cauchy problem and the initial boundary value problem. We refer the reader to Cho and Kim [1–3], Hoff [4], Hoff and Serre [5], Hoff and Smoller [6], Huang et al. [7–9], Huang and Xin [10], Zhou and Lei [11] and the references therein. The main concern is the theory of well-posedness. Nash [12] obtained local existence of classical solutions for problem (1) without vacuum. The uniqueness result is due to Serrin [13]. Recently Cho and Kim [1] established existence and uniqueness of local strong solutions with initial density vanishing in an open set. It is still a major open problem for the global existence of smooth solution to system (1) in higher space dimensions. However, some partial results have been established. Matsumura and Nishida [14] proved that global solution exists in $C^1([0,\infty), H^m(\mathbb{R}^n))$ if there exists a constant $\overline{\rho} > 0$ such that $(\rho_0 - \overline{\rho}, u_0, S_0) \in H^m(\mathbb{R}^n) \cap L^1(\mathbb{R}^n)$ and $\|\rho_0 - \rho, u_0, S_0\|_{H^m(\mathbb{R}^n)}$ is sufficiently small, with $m > [n/2] + 2$. Antontsev et al. [15] obtained the same result in the case $n = 1$ without the smallness assumption of the norm $\|\rho_0 - \rho, u_0, S_0\|_{H^m(\mathbb{R}^n)}$.

The first blow up result for compressible Navier–Stokes equation is due to Xin [16], in which the following theorem is established.

Theorem 1.1 (*[16], Theorem 1.3*)**.** *Consider the following compressible Navier–Stokes system*

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0, \\ \partial_t(\rho u) + \nabla \cdot (\rho u \otimes u) + \nabla p = \mathrm{div} T, \\ \partial_t(\rho E) + \nabla \cdot (\rho E u + p u) = \nabla \cdot (T u) + \kappa \Delta_x \theta, \end{cases} \tag{4}$$

where E, θ denote the total energy and absolute temperature respectively, T is the stress tensor given by

$$T = \mu(\nabla u + \nabla u^t) + \lambda(\nabla \cdot u) I, \tag{5}$$

and div denotes the divergence of tensor. Assume that

$$\mu > 0, \qquad \lambda + \frac{2\mu}{n} > 0, \qquad \kappa = 0, \tag{6}$$

then there is no non-trivial solution in $C^1([0,\infty), H^m(\mathbb{R}^n))$ to the Cauchy problem (4), provided that the initial density has compact support.

The proof of the theorem showed above depends crucially on the compactness assumption of support of the initial density. And it seems natural to be ill-posed when there is vacuum at the initial density, because of the physical background of the Navier–Stokes system. Later, Cho and Jin [3] generalized the blow up result to the case $\kappa > 0$ and gave a sufficient condition for blow up result provided that the initial density is positive but decays at infinity. Then Rozanova [17] established a similar blow up result with the assumption of rapidly decay instead of compact support of initial data.

Definition 1.2 (*Classical Solutions*)**.** Let T be positive. Then $(\rho(t,x), u(t,x))$ is called a classical solution to the compressible Navier–Stokes system (1) on $(0,T) \times \mathbb{R}^3$ if $\rho \in C^1([0,T) \times \mathbb{R}^3)$, $u \in C^1([0,T), C^2(\mathbb{R}^3))$, and satisfies the system (1) pointwisely on $(0,T) \times \mathbb{R}^3$.

In the present work we consider the isentropic compressible Navier–Stokes system (1) in $\mathbb{R}^3$. Blow up of classical solution in a finite time will be established under the assumption that the gradient velocity