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Global stability for a nonlocal reaction–diffusion population model

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A B S T R A C T

In this paper, we study a nonlocal reaction–diffusion population model. We establish a comparison principle and construct monotone sequences to show the existence and uniqueness of the solution to the model. We then analyze the global stability for the model.

1. Introduction

In this paper, we consider the following nonlocal reaction–diffusion population model that describes the dynamics of a single species with competition between individuals:

$$\begin{aligned} &u_t - \Delta u = u\left[f(u) - \alpha\int_{\mathbb{R}^n} g(x-y)u(y,t)dy\right], \\ &u(x,0) = u_0(x) \end{aligned} \tag{1.1}$$

for $x \in \mathbb{R}^n$ and $t \in (0,\infty)$, where $\alpha > 0$, g, u_0 are nonnegative continuous functions, and f is a continuously differentiable function on $[0,\infty)$.

Over the past few years, there have been several studies on the population models with spatial nonlocality. In [1], Britton used (1.1) with $f(u) = 1 + au - bu^2$ to model a single species that takes advantage from local aggregation (represented by au) and competes for local space (represented by $-bu^2$) and nearby resources (represented by the convolution term). Later, Britton [2] conducted a linear stability analysis for the uniform steady-state solution of (1.1) and then considered the bifurcation from the uniform steady-state. Gourley et al. [3] and Billingham [4] studied traveling wavefront solutions of (1.1) with a special form of the kernel g using numerical and asymptotic techniques. Deng [5] established the existence/uniqueness and uniform

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persistence results for (1.1), and Sun [6] proved the existence and uniqueness of positive solutions for a related nonlocal dispersal population model. On the other hand, many authors have investigated the linear stability and global stability of equilibria for specific models with nonlocal spatial effects on bounded domains under homogeneous boundary conditions (see [7–9] and their references).

Our main objective here is to conduct a global stability analysis for the uniform steady-state solution of (1.1). However, all arguments for previous models with spatial nonlocality on unbounded domains do not appear to apply. Therefore, in this paper we will adopt a different approach that is based on successive improvement of upper and lower solutions. Similar approaches have been used for models with temporal nonlocality on bounded domains [10–12] which rely on the notion of a pair of upper and lower solutions introduced by Redlinger in [13]. Because model (1.1) is on an unbounded domain, we will introduce the definition of a pair of *coupled* upper and lower solutions. The same definition was used in [5], and where a comparison principle was established by assuming $u_0 \in L^1(\mathbb{R}^n)$. Nevertheless, such an assumption cannot be imposed on the initial data in (1.1) since we require $u_0(x) \geq \delta$ for some $\delta > 0$ to carry out the stability analysis, and consequently the arguments of [5] do not apply.

The paper is organized as follows. In Section 2, we define a pair of coupled upper and lower solutions and establish a comparison principle. In Section 3, we construct two monotone sequences of upper and lower solutions and show their convergence to the unique solution of (1.1). In Section 4, we analyze the global stability for model (1.1).

2. Comparison principle

In order to conduct our discussion, we first impose the following hypotheses.

(H1) $g(x)$ is a continuous, nonnegative function on $\mathbb{R}^n$ with $g \in L^1(\mathbb{R}^n)$ and $\int_{\mathbb{R}^n} g(x)dx = 1$.
(H2) $u_0(x)$ is a continuous, nonnegative function on $\mathbb{R}^n$ with $u_0 \in L^\infty(\mathbb{R}^n)$.

For simplicity, let $D_T = \mathbb{R}^n \times (0,T)$ and $D_T \cup \Gamma_T = \mathbb{R}^n \times [0,T)$. We then introduce the following definition of coupled upper and lower solutions of problem (1.1).

Definition 2.1. A pair of functions $\overline{u}(x,t)$ and $\underline{u}(x,t)$ are called an upper and a lower solution of (1.1) on D_T, respectively, if they satisfy the following conditions.

(i) $\overline{u}, \underline{u} \in C^{2,1}(D_T) \cap C_B(D_T \cup \Gamma_T)$.
(ii) $\overline{u}(x,0) \geq u_0(x) \geq \underline{u}(x,0)$ in $\mathbb{R}^n$.
(iii) For any $(x,t) \in D_T$,

$$\overline{u}_t - \Delta\overline{u} \geq \overline{u}\left(f(\overline{u}) - \alpha\int_{\mathbb{R}^n} g(x-y)\underline{u}(y,t)dy\right) \tag{2.1}$$

$$\underline{u}_t - \Delta\underline{u} \leq \underline{u}\left(f(\underline{u}) - \alpha\int_{\mathbb{R}^n} g(x-y)\overline{u}(y,t)dy\right), \tag{2.2}$$

where $C_B(D_T \cup \Gamma_T)$ is the space of all bounded continuous functions on $D_T \cup \Gamma_T$.

Comparison principle and existence results have been established in [13] for nonlocal problems on bounded domains. However, since a continuous function may not attain its maximum or minimum value on an unbounded domain, the argument in [13] cannot be applied to problem (1.1). On the other hand, in [14] for a problem on $\mathbb{R} \times [0,\infty)$, inspired by the Phragmèn–Lindelöf Principle for uniformly parabolic operators