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A free boundary value problem modelling microelectromechanical systems with general permittivity

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ABSTRACT

An investigation of the free boundary value problem arising from the modelling of electrostatically actuated microelectromechanical systems with general permittivity is presented. Consisting of a parabolic evolution problem for the displacement of a membrane as well as of an elliptic moving boundary problem for the electric potential between the membrane and a rigid ground plate, the system is shown to be well-posed locally in time for all values λ of the applied voltage. It is in addition verified that the solution exists even globally in time, provided that the applied voltage does not exceed a certain critical value λ_*. Furthermore, we establish the convergence of the solution of the free boundary problem towards the solution of the small-aspect ratio model, as the aspect ratio tends to zero.

1. Introduction

It is the intention of this contribution to analyse qualitative properties of a parabolic free boundary value problem describing an idealised electrostatically actuated microelectromechanical system (MEMS) with general permittivity profile. Such an idealised MEMS device consists of a rigid ground plate above which a deformable membrane is suspended that is held fixed along its boundary. An application of a voltage difference between the two plates gives rise to a deflection of the membrane, meaning that a transformation of electrostatic energy into mechanical energy takes place. In the centre of attention is then in general to gain knowledge about the displacement of the membrane on the one hand and about the electrostatic potential between the two plates on the other hand.[1]

Denoting for $t > 0$ and $x \in I := (-1, 1)$ by $u = u(t, x)$, $\psi = \psi(t, x, z)$ and $f = f(x, u(t, x))$ the displacement of the membrane, the electric potential and the permittivity profile, respectively, and assuming that there is no variation of both, u and ψ, in the horizontal direction orthogonal to the x-direction, we consider

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[1] See Fig. 1 for a sketch of the considered idealised MEMS device.

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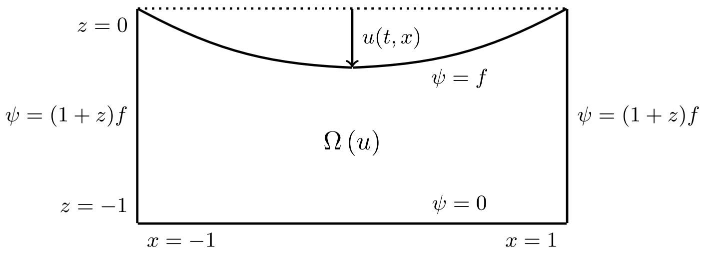

Fig. 1. Sketch of a MEMS device.

the following free boundary value problem.[2] With $\varepsilon > 0$ being the so-called *aspect ratio* of the device, i.e. the ratio of the undeformed gap size to the length of the device, the time evolution of the displacement u of the membrane is characterised by the semilinear parabolic initial boundary value problem

$$\begin{aligned} u_t - u_{xx} &= -\lambda\left(\varepsilon^2\big(\psi_x(t,x,u)\big)^2 + \big(\psi_z(t,x,u)\big)^2\right) \\ &\qquad + 2\lambda\left(\varepsilon^2\psi_x(t,x,u)f_x(x,u) + \psi_z(t,x,u)f_u(x,u)\right), \quad t>0,\ x\in I, \end{aligned} \tag{1}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{2}$$

$$u(0,x) = u_*(x), \quad x\in I, \tag{3}$$

whereas the electrostatic potential ψ is determined by the elliptic boundary value problem

$$\varepsilon^2\psi_{xx} + \psi_{zz} = 0, \quad t>0,\ (x,z)\in \Omega(u(t)), \tag{4}$$

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)}f(x,u(t,x)), \quad t>0,\ (x,z)\in\partial\Omega(u(t)), \tag{5}$$

in the region

$$\Omega(u(t)) = \{(x,z)\in(-1,1)\times(-1,\infty); -1<z<u(t,x)\} \tag{6}$$

between the rigid ground plate at $z=-1$ and the elastic membrane at $z=u(t,x)$.

The additional parameter $\lambda > 0$ occurring in the above system is proportional to the applied voltage and is thus expected to have a particular influence on the behaviour of the solution.

Observe that on the one hand the shape of the region $\Omega(u(t))$ changes with time as the membrane deflects, and on the other hand the right-hand side of the evolution equation depends on the partial derivatives of the potential, whence Eqs. (1) and (4) are strongly coupled. However, it is still a quite common approach in MEMS research to assume the aspect ratio ε of the device to be very small, i.e. to consider the problem as if the two plates were locally parallel. Formally, sending ε to zero enables one to state the solution ψ to (1)–(5) explicitly,

$$\psi(t,x,z) = \frac{1+z}{1+u(t,x)}f(x,u(t,x)), \quad t>0,\ (x,z)\in\Omega(u(t)),$$

whence in this case the displacement of the membrane is required to behave according to the so-called *small-aspect ratio model*

$$u_t - u_{xx} = -\lambda\left(\frac{f(x,u(t,x))}{1+u(t,x)}\right)^2 + 2\lambda\frac{f(x,u(t,x))}{1+u(t,x)}f_u(x,u(t,x)), \quad t>0,\ x\in I, \tag{7}$$

$$u(t,\pm 1) = 0, \quad t>0, \tag{8}$$

$$u(0,x) = u_*(x), \quad x\in I. \tag{9}$$

[2] Note that we consider the system of partial differential equations in dimensionless form. For more details on the scaling the reader is referred to Section 2.