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Well-posedness, blow-up phenomena and persistence properties for a two-component water wave system

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A B S T R A C T

We first establish the local well-posedness for the Cauchy problem of a two-component water waves system in nonhomogeneous Besov spaces using the Littlewood–Paley theory. Then, we derive three new blow-up results for strong solutions to the system. Finally, we present two persistence properties for strong solutions to the system.

1. Introduction

In this paper we consider the Cauchy problem of the following two-component water wave system [1]:

$$\begin{cases} m_t + um_x + amu_x = \alpha u_x - \kappa\rho\rho_x, & t > 0,\ x \in \mathbb{R}, \\ \rho_t + u\rho_x + (a-1)u_x\rho = 0, & t > 0,\ x \in \mathbb{R}, \\ m(0,x) = m_0(x), & x \in \mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $m = u - u_{xx}$, $a \neq 1$ is a real parameter, α is a constant which represents the vorticity of underlying flow, and $\kappa > 0$ is an arbitrary real parameter. The system is written in terms of velocity u and locally averaged density ρ.

When $\rho \equiv 0$, the system (1.1) becomes a one-component family of equations which is called the b-equation. The b-equation possess a number of structural phenomena which are shared by solutions of the family of

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equations [2–4]. Recently, some authors devoted to study the Cauchy problem of the b-equation. The local well-posedness of the b-equation on the line was obtained by Escher and Yin in [2] and by Gui et al. in [5] respectively, and on the circle by Zhang and Yin in [6]. It also has global solutions [2,5,6] and solutions which blow up in finite time [2,5,6]. The uniqueness and existence of global weak solutions to the b-equation under some certain sign conditions were obtained in [2,6].

However, there are just two members of this family which are integrable [7]: the Camassa–Holm [8,9] equation, when $a = 2$, and the Degasperis–Procesi [10] equation, when $a = 3$. The Cauchy problem and initial–boundary value problem for the Camassa–Holm equation have been studied extensively [11–18]. It has been shown that this equation is locally well-posed [11–13,16,17] for initial data $u_0 \in H^s(\mathbb{R})$, $s > \frac{3}{2}$. More interestingly, it has global strong solutions [19,11,12] and also finite time blow-up solutions [19,20,11, 12,21,13,16,17]. On the other hand, it has global weak solutions in $H^1(\mathbb{R})$ [22–26]. Finite propagation speed and persistence properties of solutions to the Camassa–Holm equation have been studied in [27,28]. After the Degasperis–Procesi equation was derived, many papers were devoted to its study, cf. [29–40].

For $a = 2$ and $\alpha = 0$, the system (1.1) becomes the two-component Camassa–Holm equation. Several types of 2-component Camassa–Holm equations have been studied in [41–51,7]. These works have established the local well-posedness [42,45–47], derived precise blow-up scenarios [45,46], and proved that there are strong solutions which blow up in finite time [42,45,47] and exist globally in time [42,47]. Moreover, it has global weak solutions [48–53].

The system (1.1) was recently introduced by Escher et al. in [1]. In [1], the authors proved the local well-posedness of (1.1) using a geometrical framework, studied the blow-up scenarios and global strong solutions of (1.1) on the circle.

However, the Cauchy problem of (1.1) on the line has not been studied yet. In this paper, using the Littlewood–Paley theory, we established the local well-posedness of (1.1) in nonhomogeneous Besov spaces which generalizes the local well-posedness result in Sobolev spaces. By Sobolev's embedding theorem, we get three new blow-up results. Furthermore, we show two persistence properties of the strong solutions to (1.1).

Our paper is organized as follows. In Section 2, we give some preliminaries which will be used in the sequel. In Section 3, we establish the local well-posedness of the Cauchy problem associated with (1.1) in Besov spaces. In Section 4, we discuss the blow-up phenomena of (1.1). In Section 5, we study the persistence properties of the strong solutions to (1.1).

Notation. In the following, we denote by $*$ the convolution. Given a Banach space Z, we denote its norm $\|\cdot\|_Z$. Since all spaces of functions are over $\mathbb{R}$, for simplicity, we drop $\mathbb{R}$ in our notations of function spaces if there is no ambiguity.

2. Preliminaries

In this section, we will recall some facts on the Littlewood–Paley decomposition, the nonhomogeneous Besov spaces and their some useful properties. We will also recall the transport equation theory, which will be used in our work. For more details, the readers can refer to [54,55].

Proposition 2.1 (*[54,55] Littlewood–Paley Decomposition*). *There exists a couple of smooth functions (χ, φ) valued in $[0, 1]$, such that χ is supported in the ball $B \triangleq \{\xi \in \mathbb{R}^n : |\xi| \le \frac{4}{3}\}$, and φ is supported in the ring $C \triangleq \{\xi \in \mathbb{R}^n : \frac{3}{4} \le |\xi| \le \frac{8}{3}\}$. Moreover,*

$$\forall\ \xi \in \mathbb{R}^n, \quad \chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1,$$

and

$$\mathrm{supp}\ \varphi(2^{-q}\cdot) \cap \mathrm{supp}\ \varphi(2^{-q'}\cdot) = \emptyset, \quad \textit{if } |q - q'| \ge 2,$$