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Approximations of steady periodic water waves in flows with constant vorticity

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ABSTRACT

We provide high-order approximations to periodic travelling wave profiles and to the velocity field and the pressure beneath the waves, in flows with constant vorticity over a flat bed.

1. Introduction

Winds in offshore storms transmit energy to the ocean surface, creating waves. Once the waves leave the storm area they become organised into two-dimensional regular wave trains, in the form of a regular profile that propagates practically unchanged in a fixed direction. These periodic travelling waves are called 'swell' in oceanography. Their shape is influenced by the underlying currents. The most significant currents on areas of the continental shelf and in many coastal inlets are the tidal currents [1]. These alternating horizontal movements of water are created by the gravitational pull of the moon, and to a lesser degree, the sun, on the earth's surface, being associated with the rise and fall of the tide: the current associated with a rising water level, called the flood, is directed towards the shore, while the current associated with a receding water level, called the ebb, is directed back out to sea. Flows of constant vorticity with a flat free surface provide adequate descriptions of pure tidal currents, cf. the discussion in [2], positive constant vorticity $\gamma > 0$ being appropriate for the modelling of the ebb current and negative constant vorticity $\gamma < 0$ for the flood current. A spectacular example of wave–current interaction is the Columbia River entrance, made by appreciable

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tidal currents one of the most hazardous navigational regions in the world since wave heights can easily be doubled in just a few hours [1]. At this location, tidal velocities in excess of 2 m/s are encountered and in winter wave heights in excess of 6 m, up to 14–15 m, are common, cf. [3]. Note that moderate tidal currents reach speeds of up to 0.7 m/s, cf. [4], while tidal currents with speeds of 5.5 m/s are encountered in between the Scottish mainland and the Orkney Islands, in the Pentland Firth—see the data provided in [5].

In this paper we provide accurate approximations for the interaction of waves with underlying currents of constant vorticity in the absence of flow-reversal; for theoretical studies and numerical simulations of wave–current interactions with flow-reversal we refer to [6–10], respectively. Let us specify that in a flow where both waves and currents are present, given velocity measurements at one point, the 'current' is defined as the average velocity, and the periodic components that vary around this average are ascribed to the wave motion. Since irrotational flows do not present flow-reversals as the underlying current must be uniform, one of our main purposes will be to highlight the effects of vorticity on the surface wave profile. This problem is of great practical relevance since the detection of non-uniform underlying currents from the surface wave pattern is particularly important to anticipate and avoid hazardous conditions, where possible. In addition to the wave profile, we will also provide approximations for the velocity profile and the pressure throughout the fluid. Not only is there a strong interplay of these flow characteristics that is very useful in qualitative studies—see [11–13], but in practice information on the state of the sea surface is often gathered from subsurface pressure and/or velocity measurements, cf. the discussions in [14–19]. Let us point out the following counter-intuitive fact: periodic travelling waves that propagate at the surface of water with a flat bed in a flow of constant vorticity must be symmetric if no flow-reversal occurs and if the wave profile is monotone between successive crests and troughs, cf. [20–22]. This means that an underlying non-uniform current of constant vorticity does not break the symmetry of irrotational wave trains, so that the manifestation of vorticity on the surface wave pattern must take subtler forms.

The fact that even when both the wave motion and the underlying current of constant vorticity are known with accuracy, their interaction produces a significantly different effect from that obtained by simply adding the effect of the waves and the currents considered separately, cf. the discussions in [23–25], shows the importance of dealing simultaneously with these two flow components. Linear theory (see the discussion in [2]) provides the dispersion relation

$$c = u_0 - \frac{\gamma \tanh(kd)}{2k} \pm \frac{1}{2k}\sqrt{\gamma^2 \tanh^2(kd) + 4gk\,\tanh(kd)} \tag{1.1}$$

for the wave speed c, expressed in terms of the surface current speed u_0 in a flow of constant vorticity γ in water of mean depth d; here $k = \frac{2\pi}{L}$ is the frequency of a wave with wavelength L. Note that (1.1) permits us to understand the effect of vorticity on the propagation speed of waves of small amplitude (that are realistically described by linear theory). For example, from the point of view of a fixed observer noticing right-propagating waves that interact with a current with vanishing surface speed (a setting corresponding to $c > 0$ and $u_0 = 0$), from (1.1) we infer that the propagation speed is

$$c(\gamma) = -\frac{\gamma \tanh(kd)}{2k} + \frac{1}{2k}\sqrt{\gamma^2 \tanh^2(kd) + 4gk\,\tanh(kd)}, \tag{1.2}$$

irrespective of the sign of γ. In this setting the underlying current beneath the flat free surface $y = 0$ and above the bed $y = -d$ is given by $u(y) = \gamma y$, so that $\gamma > 0$ corresponds to an adverse current since everywhere beneath the surface the current velocity opposes the direction of wave propagation, $\gamma < 0$ corresponds to a favourable current (the current velocity points everywhere beneath the surface in the direction of wave propagation), while the irrotational case $\gamma = 0$ is characterised by the absence of an underlying current. A simple analysis of (1.2) confirms that

$$c(-|\gamma|) > c(0) > c(|\gamma|)$$