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The Ekeland variational principle for equilibrium problems revisited and applications

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ABSTRACT

In this paper, we study the Ekeland variational principle for equilibrium problems under the setting of real Banach spaces. We make use of techniques related to infinite dimensional spaces to solve the weakly compact case and introduce a suitable weakened set of coerciveness to deal with the non weakly compact case. Some old results for the Euclidean space $\mathbb{R}^n$ are generalized under weakened conditions of semicontinuity and applications to countable rather than finite systems of equilibrium problems on real Banach spaces are derived. Applications to quasi-hemivariational inequalities are discussed.

1. Introduction

The Ekeland variational principle is a powerful tool which entails the existence of approximate solutions of minimization problems for lower semicontinuous functions on complete metric spaces. It is widely used to solve different problems of differential equations and partial differential equations, fixed point theory, optimization, mathematical programming and many other problems of nonlinear analysis. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the gradient is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool that is effective in numerous situations, which led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton–Jacobi theory, extremal problems, the Ljusternik–Schnirelmann theory, etc. The Ekeland variational principle [1] was established in 1974 and is the nonlinear version of the Bishop–Phelps theorem [2,3], with its main feature of how to use the norm completeness and a partial ordering to obtain a point where a linear functional achieves its supremum on a closed bounded convex set.

The so-called equilibrium problem is a problem of finding $x^* \in A$ such that

$$\Phi\left(x^*, y\right) \geq 0 \quad \text{for all } y \in A, \tag{EP}$$

where A is a given set and $\Phi : A \times A \longrightarrow \mathbb{R}$ is a bifunction, called equilibrium bifunction if $\Phi(x, x) = 0$, for every $x \in E$. Such a problem is also known under the name of equilibrium problem in the sense of Blum, Muu and Oettli (see [4,5]) or Ky Fan equilibrium problem (see [6]).

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It is well known that variational inequalities, mathematical programming, Nash equilibrium, optimization and many other problems arising in nonlinear analysis are special cases of equilibrium problems, see for example [7–9] and the references therein.

In this paper, the set A is not assumed to be necessarily convex which leads to an interesting case of equilibrium problems called sometimes nonconvex equilibrium problems, see [10] and the references therein.

The Ekeland variational principle for equilibrium problems has been first considered in [7,11] in the setting of the Euclidean space $\mathbb{R}^n$. It has also been the subject of study in [12] for vector equilibrium problems defined on complete metric spaces and with values in locally convex spaces ordered by closed convex cones. Latter on, several authors have been interested in such problems and several results have been obtained under various kinds of generalized metric spaces including quasi-metric spaces with different types of functions such as τ-functions and Q-functions. Many connections with fixed point theory such as the Caristi–Kirk-type fixed point theorem for multivalued mappings have been derived.

In [7,11], the authors mention that the results of the Ekeland variational principle for equilibrium problems and systems of equilibrium problems obtained on compact and on closed subsets of Euclidean spaces could be extended to reflexive Banach spaces. Motivated by this question, we establish that this is not systematic and depends also on other properties and especially, on the finite or infinite dimension of the space.

In this paper, we obtain under weakened conditions of semicontinuity an existence result for equilibrium problems defined on weakly compact subsets of real (non necessarily reflexive) Banach spaces. Then, we introduce a weakly compact set of coerciveness rather than closed balls in order to solve equilibrium problems defined on weakly closed subsets of real reflexive Banach spaces.

Instead of finite systems of equilibrium problems on the Euclidean space $\mathbb{R}^n$, we consider here countable systems of equilibrium problems defined on real Banach spaces. Then, we solve the problem when it is defined on weakly sequentially compact subsets and make use of the properties of countable product of real Banach spaces to solve the problems when it is defined on weakly closed subsets.

In the last part of this paper, we give a discussion in order to make connection between the Ekeland variational principle and fixed point theory. Such connection may be useful in certain situations and in particular, when dealing with variational inequalities. To our knowledge, there does not seem to be any result in the literature with application of the Ekeland variational principle in the analysis studies of variational inequalities. In this direction, we consider quasi-hemivariational inequalities introduced in [13]. Quasi-hemivariational inequalities are generalization of multivalued variational inequalities and several connections with equilibrium problems are obtained, see also [14–16,10] for recent and old investigations on the subject. Our approach is based on a result of [15] stating that a point is a solution of a quasi-hemivariational inequality if and only if it is a fixed point of a suitable multivalued mapping.

2. Notations and preliminaries

As it is well known, the Ekeland variational principle characterizes the completeness of a metric space and has many applications in nonlinear analysis. Several forms and variants of the Ekeland variational principle exist in the literature where the following result summarizes the relationship between this famous variational principle and the completeness of metric spaces.

Theorem 2.1. *Let (X, d) be a metric space. Then X is complete if and only if for every lower semicontinuous and lower bounded function $f : X \to \mathbb{R} \cup \{+\infty\}$ and for every $\varepsilon > 0$ there exists a point $x^* \in X$ satisfying*

$$\begin{cases} f\left(x^*\right) \le \min_{x \in X} f(x) + \varepsilon, \\ f(x) - f\left(x^*\right) + \varepsilon d\left(x^*, x\right) \ge 0, \quad \forall x \in X. \end{cases}$$

Since the problem of minimization is a special case of equilibrium problems, an extended form of the Ekeland variational principle called *Ekeland variational principle for equilibrium problems* has been first introduced in [11,7] in the setting of the Euclidean space $\mathbb{R}^n$. It is followed by several extensions under different kinds of conditions by several authors, see for instance, [17].

Here is the complete metric version of the Ekeland variational principle for equilibrium problems.

Theorem 2.2. *Let A be a nonempty closed subset of a complete metric space (X, d) and $\Phi : A \times A \to \mathbb{R}$ be a bifunction. Assume that the following conditions hold:*

(1) *$\Phi(x, x) = 0$, for every $x \in A$;*
(2) *$\Phi(z, x) \le \Phi(z, y) + \Phi(y, x)$, for every $x, y, z \in A$;*
(3) *Φ is lower bounded and lower semicontinuous in its second variable.*

Then, for every $\varepsilon > 0$ and for every $x_0 \in A$, there exists $x^* \in A$ such that

$$\begin{cases} \Phi\left(x_0, x^*\right) + \varepsilon d\left(x_0, x^*\right) \le 0, \\ \Phi\left(x^*, x\right) + \varepsilon d\left(x^*, x\right) > 0, \quad \forall x \in A,\ x \ne x^*. \end{cases}$$