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Existence of solutions for a system of coupled nonlinear stationary bi-harmonic Schrödinger equations

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ABSTRACT

We obtain existence and multiplicity results for the solutions of a class of coupled semi-linear bi-harmonic Schrödinger equations. Actually, using the classical Mountain Pass Theorem and minimization techniques, we prove the existence of critical points of the associated functional constrained on the *Nehari manifold*.

Furthermore, we show that using the so-called *fibering method* and the *Lusternik–Schnirel'man theory* there exist infinitely many solutions, actually a countable family of critical points, for such a semilinear bi-harmonic Schrödinger system under study in this work.

1. Introduction

This work is devoted to the analysis of solutions that solve the following coupled nonlinear stationary bi-harmonic Schrödinger system (BNLSS)

$$\begin{cases} \Delta^2 u_1 + \lambda_1 u_1 = \mu_1 |u_1|^{2\sigma} u_1 + \beta |u_2|^{\sigma+1} |u_1|^{\sigma-1} u_1 \\ \Delta^2 u_2 + \lambda_2 u_2 = \mu_2 |u_2|^{2\sigma} u_2 + \beta |u_1|^{\sigma+1} |u_2|^{\sigma-1} u_2 \end{cases} \tag{1.1}$$

where $\lambda_j, \mu_j > 0$, $u_j \in W^{2,2}(\mathbb{R}^N)$ with $j = 1, 2$, β denotes a real parameter and $x \in \mathbb{R}^N$, with $N = 2, 3$ (for physical purposes).

To simplify the computations in this work we assume $\sigma = 1$, hence we will study the system

$$\begin{cases} \Delta^2 u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 \\ \Delta^2 u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 \end{cases} \tag{1.2}$$

which has been analysed in the context of stability of solitons in magnetic materials when effective quasi-particle mass becomes infinite. Moreover, note that system (1.2) appears after assuming the bi-harmonic nonlinear Schrödinger equation (BNLSE) of the form

$$iW_t - \Delta^2 W + \kappa |W|^{2\sigma} W = 0, \tag{1.3}$$

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where i denotes the imaginary unit, $\kappa > 0$. Then, if $\kappa = 1$ and W is the sum of two right and left-hand polarized waves a_1W_1 and a_2W_2, where $a_1, a_2 \in \mathbb{R}$, the preceding equation (1.3) provides us with the following coupled nonlinear bi-harmonic Schrödinger system

$$\begin{cases} iW_{1,t} - \Delta^2 W_1 + |a_1W_1 + a_2W_2|^{2\sigma} W_1 = 0 \\ iW_{2,t} - \Delta^2 W_2 + |a_1W_1 + a_2W_2|^{2\sigma} W_2 = 0 \end{cases} \tag{1.4}$$

where $W_{j,t} = \frac{\partial W_j}{\partial t}$, $j = 1, 2$. For this problem we look for standing waves or finite-energy waveguide solutions of the form

$$W_j(t, x) = e^{i\lambda_j t} u_j(x), \quad j = 1, 2,$$

where $\lambda_j > 0$ and u_i are real value functions, which solve the system (1.2). Rearranging terms in (1.4) one can easily see that u_j solve the stationary system (1.2).

Problem (1.2) is the *bi-harmonic* version of a similar one studied, among others, in [1–5] where a non-linear system of coupled nonlinear Schrödinger equations (NLSE) of the form

$$\begin{cases} -\Delta u_1 + \lambda_1 u_1 = \mu_1 u_1^3 + \beta u_2^2 u_1 \\ -\Delta u_2 + \lambda_2 u_2 = \mu_2 u_2^3 + \beta u_1^2 u_2 \end{cases} \tag{1.5}$$

with direct applications to nonlinear optics, Bose–Einstein condensates, etc., was considered; see for instance [6]. See also [7] where a linearly coupled system was considered and note that in [8] system (1.5) was studied in the one-dimensional case dealing with the fractional Schrödinger operator $(-\Delta)^s + \mathrm{Id}$, $\frac{1}{4} < s \le 1$.

Here, we assume that the solutions belong to the Sobolev space $E = W^{2,2}(\mathbb{R}^N)$, endowed with the scalar product and norm

$$\langle u, v\rangle_j := \int_{\mathbb{R}^N} \Delta u \cdot \Delta v + \lambda_j \int_{\mathbb{R}^N} uv, \qquad \|u\|_j^2 = \langle u, u\rangle_j, \quad j = 1, 2. \tag{1.6}$$

Also, we define $\mathbb{E} = E \times E$, and the elements in $\mathbb{E}$ will be denoted by $\mathbf{u} = (u_1, u_2)$; as a norm in $\mathbb{E}$ we will take

$$\|\mathbf{u}\|^2 = \|u_1\|_1^2 + \|u_2\|_2^2.$$

Moreover, we denote H as the space of radially symmetric functions in E, and $\mathbb{H} = H \times H$. For $u \in E$, respectively, $\mathbf{u} \in \mathbb{E}$, we set

$$I_j(u) = \frac{1}{2} \int_{\mathbb{R}^N} \left(|\Delta u|^2 + \lambda_j u^2\right) \mathrm{d}x - \frac{1}{4} \mu_j \int_{\mathbb{R}^N} u^4 \,\mathrm{d}x, \tag{1.7}$$

$$F(\mathbf{u}) = \frac{1}{4} \int_{\mathbb{R}^N} \left(\mu_1 u_1^4 + \mu_2 u_2^4\right) \mathrm{d}x, \tag{1.8}$$

$$G(\mathbf{u}) = G(u_1, u_2) = \frac{1}{2} \int_{\mathbb{R}^N} u_1^2 u_2^2 \,\mathrm{d}x, \tag{1.9}$$

$$\mathcal{J}(\mathbf{u}) = \mathcal{J}(u_1, u_2) = I_1(u_1) + I_2(u_2) - \beta\, G(u_1, u_2) \tag{1.10}$$

$$= \frac{1}{2}\|\mathbf{u}\|^2 - F(\mathbf{u}) - \beta\, G(\mathbf{u}). \tag{1.11}$$

Remark 1.1. We recall a well known result about continuous Sobolev embedding (see, for instance, [9,10]),

$$E \hookrightarrow L^p(\mathbb{R}^N), \quad \text{with } 2 \le p \le 2^*, \tag{1.12}$$

which are compact replacing E by the radial subspace H and if in addition $N \ge 2$ and $2 < p < 2^*$ (see [10]). Besides, we recall here the definition of the critical exponent

$$2^* = \frac{2N}{N-4} \quad \text{if } N \ge 5, \quad \text{and} \quad 2^* = \infty \quad \text{for } N = 1, 2, 3, 4. \tag{1.13}$$

We observe that, by (1.12), the functional $\mathcal{J}$ is well defined since F, G make sense for $4 \le 2^* \Leftrightarrow N \le 8$, moreover, for $2 \le N < 8$ we have that F, G are compact on $\mathbb{H}$. Furthermore, it is easy to prove that the functional $\mathcal{J}$ associated to (1.2) is $\mathcal{C}^1$.