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Existence of strong solutions to the equations of unsteady motion of shear thickening incompressible fluids

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ABSTRACT

We address the existence of strong solutions to a system of equations of motion of an incompressible non-Newtonian fluid. Our aim is to prove the short-time existence of strong solutions for the case of shear thickening viscosity, which corresponds to the power law $\nu(\boldsymbol{D}) =| \boldsymbol{D}|^{q-2}$ $(2 < q < +\infty)$. In particular, we find that global strong solutions exist whenever $q > 2.23\ldots$. The results are obtained by flattening the boundary and by using the difference quotient method. Near the boundary, we use weighted estimates in the normal direction.

1. Introduction

We consider the existence of strong solutions of an incompressible fluid with shear thickening viscosity, which includes the Ostwald–de Waele model (see [1]). Ladyzhenskaya first studied such a flow in [2]. Existence of weak solutions have been proved in [3] and for the general case in [4,5].

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain. Given $0 < T < \infty$, we denote by Q_T the space time cylinder $\Omega \times]0, T[$. We consider a non-Newtonian incompressible fluid governed by the following system of PDEs:

$$\operatorname{div} \boldsymbol{u} = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} - \nabla p \quad \text{in } Q_T. \tag{1.2}$$

The notation $\boldsymbol{u} \cdot \nabla$ stands for the sum $u^i \partial_{x_i}$, where repeated subscripts and superscripts imply summation over $i = 1$ to $i = 3$. Here, $\boldsymbol{u} = (u^1, u^2, u^3)^\top$ denotes the unknown velocity of the fluid and p the pressure. Furthermore, $\boldsymbol{f} = (f^1, f^2, f^3)^\top$ denotes a given external force. In addition, in the second equation $\boldsymbol{\sigma} = (\sigma_{ij})$ denotes the deviatoric stress, which is defined by

$$\sigma_{ij} = S_{ij}(\boldsymbol{D}(\boldsymbol{u})) \quad (i, j = 1, 2, 3), \qquad \boldsymbol{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^\top).$$

Then $\mathbb{T} := \sigma - \boldsymbol{I}p$ is called the *full stress*, such that $-\operatorname{div} \mathbb{T}$ represents the sum of the internal forces due to friction, which depends mostly on the material of the fluid. In engineering practice, models of fluids with shear-rate-dependent viscosity,

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i.e., $\boldsymbol{S}(\boldsymbol{D}(\boldsymbol{u})) = \nu(|\boldsymbol{D}(\boldsymbol{u})|)\boldsymbol{D}(\boldsymbol{u})$, are frequently used. One of the most popular models among such fluids is that of the so-called power-law fluids of the following types ($1 < q < \infty$):

$$(M_1) \quad \nu(s) = \nu_0 + \nu_1 s^{q-2},$$
$$(M_2) \quad \nu(s) = \nu_2(1+s)^{q-2},$$
$$(M_3) \quad \nu(s) = \nu_3 s^{q-2},$$

where $\nu_0, \dots, \nu_3$ are constants. Here, the fluid is called *shear thinning* if $1 < q < 2$ and *shear thickening* if $2 < q < +\infty$. For more details regarding the fluid mechanical background see, e.g., [6].

From the mathematical point of view, the third case is the most difficult one, owing to its degenerate behavior, especially in the shear thickening case, $2 < q < +\infty$. Now, we impose the following conditions on $\boldsymbol{S} = (S_{ij})$ including the model (M_3):

Let $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$ denote the space of all symmetric matrices $\boldsymbol{A} = \{A_{ij}\}$ equipped with the scalar product $\boldsymbol{A} : \boldsymbol{B} = A_{ij}B_{ij}$ and norm $|\boldsymbol{A}| = \sqrt{\boldsymbol{A} : \boldsymbol{A}}$. Then, let $\boldsymbol{S} : \mathbb{M}^{3\times 3}_{\mathrm{sym}} \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ be differentiable, fulfilling the following:

(i) *Growth condition*: For all $\boldsymbol{A} \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$

$$\boldsymbol{S}(\boldsymbol{A}) \le c_1|\boldsymbol{A}|^{q-1}, \qquad |\partial_{\boldsymbol{A}}\boldsymbol{S}(\boldsymbol{A})| \le c_2|\boldsymbol{A}|^{q-2}. \tag{1.3}$$

(ii) *Coercivity*: For all $\boldsymbol{A}, \boldsymbol{B} \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$

$$\partial_{A_{kl}} S_{ij}(\boldsymbol{A})B_{ij}B_{kl} \ge c_3|\boldsymbol{A}|^{q-2}|\boldsymbol{B}|^2.$$

(iii) *Potential*: There exists $\varphi : \mathbb{M}^{3\times 3}_{\mathrm{sym}} \to \mathbb{R}$ such that, for all $\boldsymbol{A} \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$,

$$\partial_{\boldsymbol{A}}\varphi = \boldsymbol{S}, \qquad c_3|\boldsymbol{A}|^q \le \varphi(\boldsymbol{A}) \le c_4|\boldsymbol{A}|^q. \tag{1.4}$$

Here, $c_i = \mathrm{const.} > 0$ $(i = 1, \dots, 4)$.

We complete the system (1.1), (1.2) with the following boundary and initial conditions:

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \times]0, T[, \tag{1.5}$$
$$\boldsymbol{u} = \boldsymbol{u}_0 \quad \text{on } \Omega \times \{0\}. \tag{1.6}$$

The aim of this article is to show the short-time existence of strong solutions in case of shear thickening fluids, $2 < q < +\infty$, and the global existence of strong solutions for q sufficiently large, possibly strictly less than $9/4$ under the boundary condition (1.5).

In some applications it might be necessary to replace the no-slip boundary condition by other ones such as slip-type boundary conditions. For more details, refer to [7,8]. Our approach to the regularity up to the boundary is based on estimates in tangential directions and weighted estimates in normal direction. Hence, we believe our results remain true with such slip boundary conditions. However, it requires additional work, so we leave this problem for a future project.

The short-time existence of strong solutions in the shear thinning case, $1 < q < 2$, has been proved in [9] for (M_3) and in [10] for (M_2) for the problem (1.1), (1.2) with periodic boundary conditions. The existence of a global strong solution for models (M_1) and (M_2) has been proved in [3] for $q \ge \frac{9}{4}$. A regularity result for model (M_1) with a no-slip boundary condition has also been obtained in [11] for $q \ge \frac{12}{5}$. The regularity of the local-in-time weak solution for models (M_2) and (M_3) has been obtained in [8] for $q > \frac{11}{5}$. The short-time regularity of the weak solution for model (M_2) has been obtained in [12]. For $q = 2$, the result is well known.

Next, let us introduce the notion of a weak solution. We start by introducing some relevant notation and suitable function spaces. We denote the usual Sobolev spaces or Lebesgue spaces by $W^{k,s}(\Omega)$, $W^{k,s}_0(\Omega)$, or $L^s(\Omega)$ $(1 \le s \le +\infty; k \in \mathbb{N})$. Throughout the paper, bold letters indicate spaces of vector functions or vector functions. Thus, we write $\boldsymbol{W}^{1,s}(\Omega)$, $\boldsymbol{W}^{1,s}_0(\Omega)$, etc. instead of $W^{1,s}(\Omega; \mathbb{R}^N)$, $W^{1,s}_0(\Omega; \mathbb{R}^N)$, etc. $(N \in \mathbb{N}, N > 1)$.

Next, we denote by $\boldsymbol{C}^\infty_{c,\sigma}(\Omega)$ the space of all solenoidal smooth vector fields in $\mathbb{R}^3$ having its support in Ω. Then we define

$$\boldsymbol{L}^s_\sigma(\Omega) = \text{completion of } \boldsymbol{C}^\infty_{c,\sigma}(\Omega) \text{ in the norm of } \boldsymbol{L}^s(\Omega),$$
$$\boldsymbol{W}^{1,s}_{0,\sigma}(\Omega) = \text{completion of } \boldsymbol{C}^\infty_{c,\sigma}(\Omega) \text{ in the norm of } \boldsymbol{W}^{1,s}_0(\Omega).$$

Let X be a Banach space. Then by $L^s(a, b; X)$ $(a < b)$, we denote the space of all Bochner measurable functions $f : (a, b) \to X$ such that

$$\int_0^T \|f(t)\|^s_X ds < +\infty \quad \text{if } 1 \le s < \infty, \qquad \operatorname*{ess\,sup}_{t\in]0,T[} \|f(t)\|_X \quad \text{if } 1 \le s = \infty.$$

Definition 1.1. Let $\boldsymbol{u}_0 \in \boldsymbol{L}^2_\sigma(\Omega)$ and let $\boldsymbol{f} \in L^{q'}(0, T; \boldsymbol{W}^{-1,q'}(\Omega))$. We call $\boldsymbol{u} \in C^\infty_w([0, T]; \boldsymbol{L}^2_\sigma(\Omega)) \cap L^q(0, T; \boldsymbol{W}^{1,q}_{0,\sigma}(\Omega))$ a *weak solution to* (1.1)–(1.6) *with bounded energy* if