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On the solutions of the cross-coupled Camassa–Holm system

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ABSTRACT

In this paper, we study the Cauchy problem for a recently derived system of two cross-coupled Camassa–Holm equations. We firstly establish the local well-posedness result of this system in Besov spaces by using Littlewood–Paley decomposition and the transport equation theory, and then present a precise blow-up scenario for strong solutions.

1. Introduction

In this paper, we consider the following Cauchy problem for the system of two cross-coupled Camassa–Holm (CCCH) equations [1]:

$$\begin{cases} m_t + 2v_x m + v m_x = 0, & t > 0,\ x \in \mathbb{R}, \\ n_t + 2u_x n + u n_x = 0, & t > 0,\ x \in \mathbb{R}, \\ m(0,x) = m_0(x), & x \in \mathbb{R}, \\ n(0,x) = n_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

where the momenta $m = u - u_{xx}$ and $n = v - v_{xx}$. The system (1.1) was derived in [1], from a variational principle by using the Euler–Poincaré theory for symmetry reduction of the right-invariant Lagrangians on the tangent space of a Lie group. From the system (1.1), we see that the momentum m (resp. n) is transported solely by the opposite induced velocity v (resp. u), and there are obviously no terms with self-interactions. Hence this system is called as the cross-coupled system.

Obviously, for $u = v$, the CCCH system is reduced to the Camassa–Holm (CH) equation with no linear dispersion [2,3]:

$$m_t + u m_x + 2u_x m = 0, \quad m = u - u_{xx}.$$

The CH equation was first derived formally by Fokas and Fuchssteiner [3] as a bi-Hamiltonian generalization of the KdV equation. However, the great interest in the CH equation lies in the fact that Camassa and Holm [2] gave a physical derivation by using an asymptotic expansion at the so-called Camassa–Holm scaling, rather than the long-wave scaling [4,5], in the Hamiltonian of Euler's equations in the shallow water regime. It describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity [2,6]. Moreover, the CH equation could be also derived as a model for the propagation of axially symmetric waves in hyperelastic rods [7]. It has a bi-Hamiltonian structure [3,8] and is completely integrable [2,9]. The CH equation gives rise to geodesic flow of a certain invariant metric on the Bott–Virasoro group [10].

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In the past two decades, a lot of works have devoted to the study of the CH equation, because it has both solitary waves like solitons [2,11,12], which replicates a feature that is characteristic for the waves of great height-waves of largest amplitude that are exact solutions of the governing equations for water waves [13–15], and solutions which blow up in finite time as wave breaking [8,16–19] (the solution remains bounded while the slope of $u(t,x)$ becomes unbounded in finite time). The well-posedness of the CH equation has been shown in [19–21] with the initial data $u_0 \in H^s(\mathbb{R}), s > \frac{3}{2}$. In particular, Danchin [18] has dealt with the Cauchy problem of the CH equation for the initial data in the Besov space $B^s_{p,r}$, with $1 \leq p, r \leq +\infty, s > \max\{1+\frac{1}{p}, \frac{3}{2}\}$. However, the Cauchy problem of the CH equation is not locally well-posed in $H^s(\mathbb{R}), s < \frac{3}{2}$. Indeed, the solution cannot depend uniformly continuously with respect to the initial data [20]. On the other hand, the CH equation has the peaked solitons (peakons) of the form $\varphi_c(t,x) = ce^{-|x-ct|}$ with the traveling speed $c > 0$. Constantin and Strauss [11] gave an impressive proof of stability of peakons by using the conservation laws. It is worthwhile to mention that there are many other integrable multi-component generalizations of the CH equation with applications in hydrodynamics, cf. [22–29].

Although the system (1.1) admits peakon solutions, it does not belong to the two-component CH integrable hierarchy. The integrability of this system is still an open problem [30]. The only known conservation laws for the system (1.1) are

$$H(u,v) = \int_{\mathbb{R}} (uv + u_x v_x)dx \quad \text{and} \quad M(u,v) = \int_{\mathbb{R}} (u+v)dx.$$

The singular solutions of waltzing peakons and compacton pairs for the system (1.1) have been studied in [1]. Applying the same momentum map as the CH equation to the system (1.1), it follows that

$$m(t,x) = \sum_{a=1}^{M} m_a(t)\delta(x-q_a(t)) \quad \text{and} \quad n(t,x) = \sum_{b=1}^{N} n_b(t)\delta(x-r_b(t)).$$

For the simplest case $M = N = 1$, when the initial datum m_0 and n_0 have the same signature, one can observe that the half period of "waltzing" motion, and two types of peakons (because of no self-interactions) exchange momentum amplitudes over a half cycle [1]. Henry, Holm and Ivanov [31] have examined whether the solutions m, n and in turn u, v of the system (1.1), which initially have compact support, would possess the same persistence properties or not. Very recently, we know that the local well-posedness for the system (1.1) with initial data in $H^s(\mathbb{R}) \times H^s(\mathbb{R}), s > \frac{5}{2}$, has been proved in [32] by Kato's semigroup theory.

The purpose of our present paper is to prove the local well-posedness of the Cauchy problem (1.1) in the nonhomogeneous Besov spaces. The method used here follows the study for the CH equation [18], but a cross-coupled two-component system of equations, instead of a scalar equation, is considered. Unlike the CH case, we need to solve transport equations satisfied by m, n, rather than u, v. Thus, our locally well-posed result requires higher regularity on the initial data $(u_0, v_0) \in H^s(\mathbb{R}) \times H^s(\mathbb{R})$ with $s > \frac{5}{2}$. Then, we firstly present a precise blow-up scenario with respect to momenta m, n (see Corollary 4.1) using the transport equation theory, and then derive a blow-up criterion for the strong solution (m, n) of the system (1.1) with respect to u, v (see Theorem 4.2). Thus, the approach in our paper is different from [32]. However, we have not found method to give some sufficient conditions on the initial data for the breaking of waves to the system (1.1). The main difficulty is that we cannot directly obtain the boundedness of the quantities $\|u\|_{L^\infty(\mathbb{R})}$ and $\|v\|_{L^\infty(\mathbb{R})}$ by using the conservation laws, in contrast to the CH equation with the conserved quantity $\|u\|_{H^1(\mathbb{R})} = \|u_0\|_{H^1(\mathbb{R})}$.

The entire paper is organized as follows. In Section 2, we present some facts on Littlewood–Paley decomposition, Besov spaces and the transport equation theory. In Section 3, we establish the local well-posedness result for the system (1.1). In Section 4, we derive a blow-up scenario of strong solutions to the system (1.1).

Notation. Since our discussion about the system (1.1) is all on the line $\mathbb{R}$, for simplicity, we omit $\mathbb{R}$ in our notations of function spaces. We denote the Fourier transform of a function u as $\mathcal{F}u$.

2. Preliminaries

In this section, we will recall some basic theory of the Littlewood–Paley decomposition and the transport equation theory on Besov spaces, which will play an important role in the sequel. One may get more details from [18,33].

Proposition 2.1 ([33] *Littlewood–Paley Decomposition*). *Let* $\mathcal{B} := \{\xi \in \mathbb{R}, |\xi| \leq \frac{4}{3}\}$ *and* $\mathcal{C} := \{\xi \in \mathbb{R}, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. *Then there exist* $\psi(\xi) \in C_c^\infty(\mathcal{B})$ *and* $\varphi(\xi) \in C_c^\infty(\mathcal{C})$ *such that*

$$\psi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}$$

and

$$\mathrm{Supp}\,\varphi(2^{-q}\cdot) \cap \mathrm{Supp}\,\varphi(2^{-q'}\cdot) = \emptyset, \quad \text{if } |q-q'| \geq 2,$$
$$\mathrm{Supp}\,\psi(\cdot) \cap \mathrm{Supp}\,\varphi(2^{-q}\cdot) = \emptyset, \quad \text{if } q \geq 1.$$