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Traveling waves solutions to general isothermal diffusion systems

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A B S T R A C T

This article studies propagating traveling waves in a class of reaction–diffusion systems in one dimensional space which model isothermal diffusion, bio-reactor and auto-catalytic chemical reactions. In addition, the reaction terms are non-KPP, which denies a linear theory for determining minimum speed. By using novel techniques, it is shown that the minimum speed depends, for a given reaction term, on the size of ratio of diffusion coefficients D in a subtle way; with the case $D > 1$ demonstrating very different properties from that of $D < 1$.

1. Introduction

In this paper we study reaction–diffusion systems of the form

$$\begin{cases} u_t = u_{xx} - H(u,v), \\ v_t = Dv_{xx} + H(u,v), \end{cases} \tag{1.1}$$

where H is a non-negative C^1 function defined on $[0,\infty) \times [0,\infty)$, $D > 0$ is a positive number. We assume $H(u,v)$ satisfies the following conditions.

(A1) $H(u,v) > 0$ on $(0,1) \times (0,1)$, $H(0,1) = H(1,0) = 0$.

The particular feature we are interested in is the existence and non-existence of traveling waves connecting the two equilibrium points $(u,v) = (0,1)$ and $(u,v) = (1,0)$. Let $u(x,t) = u(z)$, $v(x,t) = v(z)$, with

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$z = x - ct,\ c > 0$, we arrive at the traveling wave problem: (u, v) is a positive solution of

$$(TW)\begin{cases} u''(z) + cu'(z) - H(u,v) = 0, & -\infty < z < \infty, \\ Dv''(z) + cv'(z) + H(u,v) = 0, & -\infty < z < \infty, \\ \lim\limits_{z\to-\infty}(u(z),v(z)) = (0,1), & \lim\limits_{z\to\infty}(u(z),v(z)) = (1,0), \\ \lim\limits_{z\to\pm\infty} u'(z) = \lim\limits_{z\to\pm\infty} v'(z) = 0, \end{cases} \tag{1.2}$$

where the positive constant c is the wave speed.

Many important phenomena in applications such as isothermal diffusion, population dynamics, bio-reactors and chemical reactions can be modeled by a system of the form as in (1.1). Since we are only interested in developing a mathematical theory on traveling waves, we refer the interested reader to [1–10] for the modeling aspects.

The main purpose of this work is to demonstrate that a number of special cases being studied in the literature can be united into a general case with minimum assumption on the nonlinearity H, thus revealing that results established for special cases are valid for more general systems. Moreover, we study some new systems which are totally different from what appears in the literature.

It is clear by adding the two equations in (TW) that $u''(z) + Dv''(z) + c(u'(z) + v'(z)) = 0$, and by integration on $(-\infty, z)$,

$$u'(z) + Dv'(z) + c(u(z) + v(z) - 1) = 0. \tag{1.3}$$

In particular, if $D = 1$, we obtain that $u(z) + v(z) \equiv 1$ and the problem reduces to a single equation

$$v''(z) + cv'(z) + F(v) = 0, \tag{1.4}$$

where $F(v) = H(1 - v, v) > 0$ on $(0, 1)$, $F(0) = F(1) = 0$, and $F'(0) \geq 0 \geq F'(1)$. By classical theory, see [11,12,10], there is a minimum speed $c_1 > 0$ such that traveling wave exists for any $c \geq c_1$.

One important aspect of our approach is to link the general case of $D > 0$ to the special case of $D = 1$ using comparison argument, where previous studies of (TW) with general H fail to explore.

The present paper concentrates on two classes of nonlinearity H with two very different behavior about $(0, 1)$: (i) $H_u(0, 1) > 0$ but $H_v(0, 1) = 0$ and (ii) $H(u, v) \approx u^\alpha v^\beta$ with $\alpha, \beta > 1$. The case of $H_u(0, 1) > 0$ and $H_v(0, 1) = 0$ put previous works in the literature on the case of $H(u, v) = uv^n$ with $n > 1$ and $H(u, v) = uv^n/(1 + u)$ (see [2,13]) into a more general framework. The case where both $H_u(0, 1) = 0$ and $H_v(0, 1) = 0$ is completely different and unknown. Our study of the case of $H(u, v) \approx u^\alpha v^\beta$ with $\alpha,\ \beta > 1$ reveals some very interesting phenomena.

Theorem 1. *Let $D > 1$. Suppose $H(u, v)$ is a C^2 function which satisfies the following conditions:*

(i) *Assumption (A1), and $H_u(0, 1) > 0$ and $H_v(0, 1) = 0$;*
(ii) *$H(u, v) \leq H_u(0, 1)u$, $H(u, v)$ is an increasing function of u on $(0, D)$, for any $v \in (0, 1)$ fixed, and $H(\alpha u, v) \leq \lambda_0 \alpha H(u, v)$ on $(0, 1) \times (0, 1)$ for any $0 < \alpha < D$, where $\lambda_0 > 0$ is a fixed number;*
(iii) *$X(c)^T M X(c) < 0$ for any $c > 0$, where M is the Hessian matrix of H at $(0, 1)$ and*

$$X(c) = \left[\left(\frac{2}{c + \sqrt{c^2 + 4H_u(0,1)}}\right)^2,\ -H_u(0,1)\right].$$

Then, there exists a unique traveling wave (up to translation) for any

$$c \geq c_1 \frac{\sqrt{D\lambda_0}}{\sqrt{1 - \left(1 - \frac{1}{D}\right)\frac{L-1}{L+1}}},$$