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Conservation laws for a modified lubrication equation

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ABSTRACT

In this work we study the conservation laws of a modified lubrication equation, which describes the dynamics of the interfacial motion in phase transition. We show that the equation is nonlinear self-adjoint and has an exact Lagrangian with an auxiliary function. As a result, by a general theorem on conservation laws proved by Nail Ibragimov recently and Noether's theorem, some new conservation laws for the equation are obtained. Our results show that the non-locally defined conservation laws generated by Noether's theorem are equivalent to the local ones given by Ibragimov's theorem.

1. Introduction

Fourth order equations arise naturally in many physical models and, among these, the lubrication equation or the corresponding modified version plays an important role in the study of the interface movements, which describes the dynamics of the interfacial motion in phase transition [1]. In this paper we will study the following modified lubricated equation (MLE) from [2],

$$u_t + f(u)u_{xxxx} = 0, \tag{1}$$

where $u(x,t)$ denotes the thickness of Hele-Shaw flow, see [3]. There have been abundant literatures which have contributed to the studies of Eq. (1) with some special thickness function $f(u)$. In [2], Bertozzi showed numerically that a solution of (1) with $f(u) = u^n$ has either finite time or infinite time singularity; Hui studied the existence of nonnegative solution of (1) with some boundary condition [4]. Recently, Gandarias, Bruzón and some authors studied the symmetries and conservation laws for Eq. (1) (see [5–9] and references therein). Noting that Eq. (1) is a scalar evolution differential equation and it does not have a classical Lagrangian, the authors in [7,8] proved that Eq. (1) is self-adjoint if and only if $f(u) = a/u$, and based on the concept of nonlinear self-adjointness the authors constructed the conservation laws by a general theorem on conservation laws proved by Ibragimov [10], which does not require the existence of a Lagrangian.

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There are many methods of constructing conservation laws for differential equations; among them the most famous theorem is Noether's theorem. It allows one to construct conservation laws for differential equations following a straightforward algorithm. Although Noether's theorem provides an elegant approach to find conservation laws, it possesses a strong limitation: it can only be applied to equations with variational structure. However a large number of differential equations without variational structure admit conservation laws, for example, Eq. (1). Thus, many authors developed some methods which do not rely on the knowledge of Lagrangian functions to obtain conservation laws, such as characteristic method [11], direct method [12,13], method of multipliers, variational approach on space of solutions of a differential equation [12,13] and so on. Especially, it is worth pointing out that the authors presented an alternative method to compute the conservation laws for an evolution equation by cosymmetry in [14]. Recently, Ibragimov has proved a result in [10] which allows one to construct conservation laws for equations without variational structure. Essentially, Ibragimov's theorem is an extension of Noether's theorem by introducing formal Lagrangian to get rid of the variational limitation.

In this paper, we consider Eq. (1) and focus on the nonlinear self-adjointness, Lagrangian and associated conservation laws for Eq. (1). On the one hand, we study the nonlinear self-adjointness of Eq. (1) and obtain a different substitution $v = v(x, u)$ from the ones obtained in [7,8]. Consequently, we can construct some new conservation laws of this equation by Ibragimov's method. On the other hand, we present an *auxiliary Lagrangian method* to construct conservation laws for Eq. (1). Based on a simple calculation, we can find an exact Lagrangian of Eq. (1) with an auxiliary function, which allows us to construct conservation laws of (1) by Noether's theorem. Comparing the conservation laws obtained by the exact Lagrangian to the ones generated by Ibragimov's theorem, we can see that they are equivalent. Our results show that Ibragimov's theorem on conservation laws is one elegant way to establish conservation laws for the equations under consideration.

For the sake of completeness, we briefly present the notations, definition of nonlinear self-adjointness, Ibragimov's theorem and Noether's theorem on conservation laws. Consider a s-th order nonlinear equation

$$E(x, u, u_{(1)}, u_{(2)}, \dots, u_{(s)}) = 0 \tag{2}$$

with n independent variables $x = (x_1, x_2, \dots, x_n)$ and a dependent variable $u = u(x)$, where $u_{(s)} = \partial^s u$. We denote a symmetry generator of (2) by

$$X = \xi^i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}, \tag{3}$$

and let

$$E^*(x, u, v, u_{(1)}, v_{(1)}, \dots, u_{(s)}, v_{(s)}) := \frac{\delta \mathcal{L}}{\delta u} = 0 \tag{4}$$

be the *adjoint equation* of Eq. (2), where $\mathcal{L} = vE$ is called *formal Lagrangian*, $v = v(x)$ is a new dependent variable and

$$\frac{\delta}{\delta u} = \frac{\partial}{\partial u} + \sum_{m=1}^{s} (-1)^m D_{i_1} \cdots D_{i_m} \frac{\partial}{\partial u_{i_1 \cdots i_m}}$$

denotes the Euler–Lagrange operator.

Now let us state the definition of nonlinear self-adjointness for a equation, see [15–20] and references therein.

Definition 1. Eq. (2) is said to be *nonlinearly self-adjoint* if the equation obtained from the adjoint equation (4) by the substitution $v = \phi(x, u)$ with a certain function $\phi(x, u) \neq 0$ is identical with the original Eq. (2). In other words, the following equation holds:

$$E^*|_{v=\phi} = \lambda(x, u, u_{(1)}, \dots) E \tag{5}$$

for some differential function $\lambda = \lambda(x, u, u_{(1)}, \dots)$.