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On a Zoltán Boros' problem connected with polynomial-like iterative equations

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ABSTRACT

We determine all continuous solutions $g\colon I \to I$ of the polynomial-like iterative equation $g^3(x) = 3g(x) - 2x$, where $I \subset \mathbb{R}$ is an interval. In particular, we obtain an answer to a problem posed by Zoltán Boros (during the Fiftieth International Symposium on Functional Equations, 2012) of determining all continuous functions $f\colon (0,+\infty) \to (0,+\infty)$ satisfying $f^3(x) = \frac{[f(x)]^3}{x^2}$.

1. Introduction

Given an interval $I \subset \mathbb{R}$ we are interested in determining all continuous functions $g\colon I \to I$ satisfying

$$g^3(x) = 3g(x) - 2x. \tag{1.1}$$

Here and throughout the paper g^n denotes the n-th iterate of a given self-mapping $g\colon I \to I$; i.e., $g^0 = \mathrm{id}_I$ and $g^k = g \circ g^{k-1}$ for all integers $k \geq 1$.

There are two reasons to find all continuous solutions $g\colon I \to I$ of Eq. (1.1). The first one is to answer a problem posed by Zoltán Boros (see [1]) of determining all continuous functions $f\colon (0,+\infty) \to (0,+\infty)$ satisfying

$$f^3(x) = \frac{[f(x)]^3}{x^2}. \tag{1.2}$$

The second reason is that Eq. (1.1) belongs to the class of important and intensively investigated iterative functional equations; i.e., the class of polynomial-like iterative equations of the form

$$\sum_{n=0}^{N} a_n g^n(x) = F(x), \tag{1.3}$$

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where a_n's are given real numbers, $F: I \to I$ is a given function and $g: I \to I$ is the unknown function. For the theory of Eq. (1.3) and its generalizations we refer the readers to books [2,3], surveys [4,5], and some recent papers [6–17].

Eq. (1.3) represents a linear dependence of iterates of the unknown function and looks like a linear ordinary differential equation with constant coefficients, expressing the linear dependence of derivatives of the unknown function. The difference between these two equations is that linear ordinary differential equations with constant coefficients have a complete theory for finding their solutions, in contrast, even to a very interesting subclass of homogeneous polynomial-like iterative equations of the form

$$\sum_{n=0}^{N} a_n g^n(x) = 0. \tag{1.4}$$

The difficulties in solving Eq. (1.4), and hence also Eq. (1.3), comes from the fact that the iteration operator $g \to g^n$ is nonlinear.

The problem of finding all continuous solutions of Eq. (1.4) for a given positive integer N seems to be very difficult. It is completely solved in [18] (see also [19]) for $N = 2$, but it is still open even in the case where $N = 3$ (see [20]). It turns out that the nature of continuous solutions of Eq. (1.3) depends deeply on the behavior of complex roots of its characteristic equation

$$\sum_{n=0}^{N} \alpha_n r^n = 0. \tag{1.5}$$

This characteristic equation is motivated by Euler's idea for differential equations; it is obtained by putting $g(x) = rx$ into (1.4) to determine all its linear solutions. There are some results describing all continuous solutions of Eq. (1.4) with $N \geq 3$ in very particular cases where the complex roots of Eq. (1.5) fulfill special conditions (see [21–23]). Note that the characteristic equation of Eq. (1.1) is of the form

$$r^3 - 3r + 2 = 0,$$

and it has two roots: $r_1 = 1$ of multiplicity 2 and $r_2 = -2$ of multiplicity 1. Therefore, none of known results can be used to determine all continuous solutions $g: I \to I$ of Eq. (1.1).

2. Preliminary

It is easy to check that the identity function, defined on an arbitrary set $A \subset \mathbb{R}$, is a continuous solution of Eq. (1.1). Thus Eq. (1.1) has the unique solution $g: I \to I$ in the case where $I \subset \mathbb{R}$ is an interval degenerated to a single point. Therefore, from now on, fix a non-degenerated interval $I \subset \mathbb{R}$; open or closed or closed on one side, possible infinite.

Lemma 2.1. *Assume that $g: I \to I$ is a continuous solution of Eq. (1.1). Then g is strictly monotone. Moreover, if $I \neq \mathbb{R}$, then g is strictly increasing.*

Proof. Fix $x, y \in I$ and assume that $g(x) = g(y)$. Then by (1.1) we obtain

$$x = \frac{3g(x) - g^3(x)}{2} = \frac{3g(y) - g^3(y)}{2} = y.$$

Since g is continuous, it follows that it is strictly monotone.

Assume now that $I \neq \mathbb{R}$ and suppose that, contrary to our claim, g is strictly decreasing. Put $a = \inf I$ and $b = \sup I$. If $b = +\infty$, then

$$-\infty < a \leq \lim_{x\to b} g^3(x) = \lim_{x\to b}\bigl(3g(x) - 2x\bigr) = -\infty,$$