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Barrier solutions of elliptic variational inequalities

Siegfried Carl

Institut für Mathematik, Martin-Luther-Universität Halle-Wittenberg, D-06099 Halle, Germany

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ABSTRACT

We consider elliptic variational inequalities in a bounded domain $\Omega \subset \mathbb{R}^N$ of the form

$$u \in K : \langle Au + \lambda |u|^{p-2}u + F(u), v - u\rangle \geq \langle h, v - u\rangle, \quad \forall\, v \in K,$$

where A is a second order quasilinear elliptic operator of divergence type, F is the operator generated by lower order terms, and K is a closed convex subset of the Sobolev space $W^{1,p}(\Omega)$, $1 < p < \infty$, and $\lambda \geq 0$. The main goal of this paper is to answer the following question: Does the variational inequality possess *barrier solutions*? Here, solutions u_* and u^* of the variational inequality are called barrier solutions if $u_* \leq u^*$, and any solution u of the variational inequality satisfies $u_* \leq u \leq u^*$. In other words, we are going to provide sufficient conditions which ensure that the solution set $\mathcal{S}$ of the variational inequality is nonempty, and $\mathcal{S}$ possesses a greatest and a smallest element with respect to the natural partial ordering of functions. An answer to the raised question is by no means trivial as will be seen by specific examples of the variational inequality under consideration. The obtained results are finally used to derive a priori order bounds as well as existence of barrier solutions for multi-valued variational inequalities including variational–hemivariational inequalities as special case.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain with Lipschitz boundary $\partial\Omega$. For $N = 1$, we assume that Ω is a bounded open interval. Let $V = W^{1,p}(\Omega)$, $1 < p < \infty$, denote the usual Sobolev space with its dual space V^*, and let $K \subset V$ be a closed, convex subset. In this paper we consider the following elliptic variational inequality: Find $u \in K$ such that

$$\langle Au + \lambda |u|^{p-2}u + F(u), v - u\rangle \geq \langle h, v - u\rangle, \quad \forall\, v \in K, \tag{1.1}$$

where A is a second order quasilinear elliptic differential operator in divergence form

$$Au(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, \nabla u(x)), \quad \text{with } \nabla u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_N}\right), \tag{1.2}$$

E-mail address: siegfried.carl@mathematik.uni-halle.de.

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F denotes the Nemytskij operator generated by a Carathéodory function $f : \Omega \times \mathbb{R} \to \mathbb{R}$, i.e., $F(u)(x) = f(x, u(x))$, $\lambda \geq 0$, and $h \in V^*$.

The main goal of this paper is to provide sufficient conditions that ensure the existence of so called *barrier solutions* of the variational inequality (1.1) whose definition is given as follows.

Definition 1.1. Solutions u_* and u^* of the variational inequality (1.1) are called **barrier solutions** of (1.1) provided that $u_* \leq u^*$, and any solution u of (1.1) satisfies $u_* \leq u \leq u^*$.

Remark 1.1. Apparently, the existence of barrier solutions requires that the set $\mathcal{S}$ of all solutions of the variational inequality (1.1) is nonempty, and that $\mathcal{S}$ possesses a greatest solution u^* and a smallest solution u_*, where the notion *greatest* and *smallest* refer to the underlying natural partial ordering $\leq$ of functions, i.e., $u \leq v$ if and only if $u(x) \leq v(x)$ for a.a. $x \in \Omega$. For example, in the trivial case that (1.1) has a uniquely defined solution u, then $u_* = u^* = u$. However, in general there may exist multiple solutions or even infinitely many solutions of (1.1), in which case the existence of barrier solutions is not at all a straightforward task.

In a number of recent papers by the author (see, e.g., [1–3]) roughly speaking the following has been shown even in a more general setting than (1.1): Given an ordered pair of appropriately defined sub- and supersolutions $\underline{u}$ and $\overline{u}$, respectively, then there exist greatest and smallest solutions within the order interval $[\underline{u}, \overline{u}]$. In this case, however, further solutions may exist which are not within the interval $[\underline{u}, \overline{u}]$, and nothing can be said about barrier solutions of the problem under consideration. Unlike in the above mentioned papers, the main goal of this paper is to provide conditions that ensure greatest and smallest solutions of the problem under consideration without assuming the existence of sub-supersolutions. The following examples of the variational inequality (1.1) show that even in simple special cases existence as well as nonexistence of barrier solutions may occur.

Example 1.1. Let $A = -\Delta$ be the negative Laplacian, $K = W_0^{1,2}(\Omega)$ the Sobolev space of functions having homogeneous generalized boundary values, and let $\lambda_1 > 0$ be the first eigenvalue of the negative Laplacian. Consider the following Dirichlet boundary value problem:

$$-\Delta u - \lambda_1 u = 0 \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega. \tag{1.3}$$

Problem (1.3) is apparently a special case of (1.1) when $p = 2$, $\lambda = 0$, and $f(x, u) = -\lambda_1 u$. It is easily seen that the set of all solutions $\mathcal{S}$ is the one-dimensional subspace given by

$$\mathcal{S} = \{u = c\varphi_1 : c \in \mathbb{R}\},$$

where φ_1 is the eigenfunction belonging to the first eigenvalue λ_1. Obviously, $\mathcal{S} \neq \emptyset$ is neither norm bounded nor does it possess barrier solutions.

Example 1.2. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and increasing with $f(0) = 0$, then the following special case of (1.1)

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega, \; u = 0 \text{ on } \partial\Omega \tag{1.4}$$

has the unique solution $u = 0$, and thus $\mathcal{S} = \{0\}$, i.e., problem (1.4) possesses barrier solutions $u_* = u^* = 0$.

Example 1.3. In [4] (see also [5]) the exact number of positive solutions of the following two-point boundary value problem involving concave and convex nonlinearities has been obtained:

$$-v'' = \mu\,(v^q + v^p) \quad \text{in } (a, b), \; v > 0 \; \text{ in } (a, b), \; v(a) = v(b) = 0, \tag{1.5}$$

where $\mu > 0$ is a parameter, and $0 < q < 1 < p$. Regarding the existence and number of positive solutions the following result has been obtained, see [4, Theorem 1]: There is a number μ^* with $0 < \mu^* < \infty$ such that