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Existence and multiplicity of solutions for asymptotically linear Schrödinger–Kirchhoff equations☆

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ABSTRACT

The purpose of this work is to study a Schrödinger–Kirchhoff equation in $\mathbb{R}^3$ with the nonlinearity asymptotically linear and the potential indefinite in sign. By variational methods, we obtain the existence of multiple nontrivial solutions for this problem.

1. Introduction and main results

In this work, we consider the following Schrödinger–Kirchhoff type problem:

$$-\left(a+b\int_{\mathbb{R}^3}|\nabla u|^2\,\mathrm{d}x\right)\Delta u+V(x)u=f(x,u),\quad \text{in } \mathbb{R}^3, \tag{1.1}$$

where $a>0$, $b\geqslant 0$ are constants.

(1.1) is an important nonlocal quasilinear problem. If $V(x)\equiv 0$ and $\mathbb{R}^3$ is replaced by a bounded domain $\Omega\subset\mathbb{R}^N$, problem (1.1) reduces to the following Dirichlet problem:

$$\begin{cases}-\left(a+b\displaystyle\int_{\Omega}|\nabla u|^2\,\mathrm{d}x\right)\Delta u=f(x,u) & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega,\end{cases} \tag{1.2}$$

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which is a generalization of a model first introduced by Kirchhoff [1]. More precisely, problem (1.2) is related to the stationary analogue of the equation

$$u_{tt}-\left(a+b\int_{\Omega}|\nabla u|^2\,\mathrm{d}x\right)\Delta u=f(x,u), \tag{1.3}$$

which is an extension of classical D'Alembert's wave equation for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Problem (1.3) has received much attention after Lions [2] proposed an abstract framework to the problem. Some early studies can be found in [3–5] and the references therein.

More recently, problems like type (1.2) (in bounded domain) have been investigated by many authors (cf., e.g., [6–13]). In [7], Ma and Muñoz Rivera obtained positive solutions via variational methods; In [8], Perera and Zhang obtained a nontrivial solution via Yang index and critical group; Zhang and Perera [9], Mao and Zhang [10] obtained multiple and sign-changing solutions via the invariant sets of descent flow; He and Zou [11,12] obtained infinitely many solutions via the local minimum methods and the fountain theorems.

Equations of type (1.1) in the whole space $\mathbb{R}^N$ have also been studied extensively; see, for example, [14–22] and the references therein. In all above studies, we notice that the potential was assumed to be radially symmetric or equipped with some "compact" condition or positive definite. In the present paper, we are going to study the existence and multiplicity of nontrivial solutions for problem (1.1) by means of Morse theory and local linking, which are different from the literature mentioned above. Before stating our main results, we need to describe the eigenvalue of the Schrödinger operator $-a\Delta+V$ first:

Consider the increasing sequence $\lambda_1\leqslant\lambda_2\leqslant\cdots$ of minimax values defined as

$$\lambda_n=\inf_{S\in\mathfrak{S}_n}\sup_{u\in S\setminus\{\mathbf{0}\}}\frac{\int_{\mathbb{R}^3}\left(a|\nabla u|^2+V(x)u^2\right)\mathrm{d}x}{\int_{\mathbb{R}^3}u^2\mathrm{d}x}$$

where $\mathfrak{S}_n$ is the family of n-dimensional subspaces of $C_0^\infty(\mathbb{R}^3)$. Remind that $a\neq 0$. Let

$$\lambda_\infty:=\lim_{k\to\infty}\lambda_n.$$

It is known that λ_∞, if finite, is the bottom of the essential spectrum of $-a\Delta+V$. Hence λ_n is an eigenvalue of $-a\Delta+V$ of finite multiplicity whenever $\lambda_n<\lambda_\infty$ (see [23,24] for details). Note that if V is bounded from below, then λ_n is well defined and is finite.

Set $F(x,u)=\int_0^u f(x,t)\,\mathrm{d}t$. We make the following assumptions:

(V) $V\in C(\mathbb{R}^3)$ bounded from below and there exists an integer $k\geqslant 1$ such that $\lambda_k<0<\lambda_{k+1}$.

(f_1) $f\in C^1(\mathbb{R}^3\times\mathbb{R})$ and there exist constants $p\in(2,6)$ and $c>0$ such that

$$|f(x,t)|\leqslant c(1+|t|^{p-1}),\quad \forall x\in\mathbb{R}^3 \text{ and } t\in\mathbb{R}.$$

(f_2) $f(x,t)=o(t)$ as $t\to 0$ uniformly in $x\in\mathbb{R}^3$.

(f_3) There exists $0<h<\lambda_\infty$ such that $F(x,t)\leqslant\frac{1}{2}ht^2$ for all $x\in\mathbb{R}^3$ and $t\in\mathbb{R}$.

(f_3') There exists $0<h<\lambda_\infty$ such that $tf(x,t)\leqslant ht^2$ for all $x\in\mathbb{R}^3$ and $t\in\mathbb{R}$.

(f_4) $f(x,-t)=-f(x,t)$.

Our main results read as follows:

Theorem 1.1. *Suppose that assumptions* (V), (f_1)–(f_3) *are satisfied, then problem* (1.1) *has at least one nontrivial solution.*

Theorem 1.2. *Suppose that assumptions* (V), (f_1), (f_2) *and* (f_3') *are satisfied, then problem* (1.1) *has at least two nontrivial solutions.*