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Solutions for a quasilinear elliptic equation in Musielak–Sobolev spaces

Duchao Liu*, Peihao Zhao

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, PR China

A R T I C L E I N F O

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Dedicated to Professor Xianling Fan on the occasion of his 70th birthday

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A B S T R A C T

Critical point theory is used to show the existence of weak solutions to a quasilinear elliptic differential equation under the functional framework of the Musielak–Sobolev spaces in a bounded smooth domain with Dirichlet boundary condition.

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1. Introduction

In this paper we deal with the existence of weak solutions to a differential equation in the Musielak–Sobolev spaces of the form

$$\begin{cases} -\mathrm{div}\big(b(x,|\nabla u|)\nabla u\big) = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain.

In the study of nonlinear differential equations, it is well known that more general functional space can handle differential equations with more nonlinearities. If we want to study a general form of differential equations, it is very important to find a proper functional space corresponding to their solutions.

Differential equations in variable exponent Sobolev spaces and Orlicz–Sobolev spaces have been studied extensively in recent years, see [1–4]. To our best knowledge, however, differential equations in

* Corresponding author. Tel.: +86 13893289235; fax: +86 09318912481.
E-mail addresses: liuduchao@gmail.com (D. Liu), zhaoph@lzu.edu.cn (P. Zhao).

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Musielak–Sobolev spaces have been studied little. In the research of the paper [5], Benkirane and Sidi give an existence result for nonlinear elliptic equations in Musielak–Sobolev spaces, and in this paper the authors give an embedding theorem in Musielak–Sobolev spaces. In two recent papers (see [6,7]), Fan gives some properties about this kind of functional space. It is possible to make an application of those properties to give some existence of solutions to the elliptic equations with much more nonlinearities. Our aim in this paper is to study the existence of solutions to a kind of elliptic differential equation under the functional frame work of Musielak–Sobolev spaces, which is a more general case of the variable exponent Sobolev spaces and Orlicz–Sobolev spaces. The method we used here is the variational method.

The paper is organized as follows. In Section 2, for the readers convenience we recall some definitions and properties about Musielak–Orlicz–Sobolev spaces. In Section 3, we give some basic properties of the differential operator in Musielak–Sobolev spaces corresponding to the equation. In Section 4, we give several existence results for the weak solutions to our problem (1.1).

2. The Musielak–Orlicz–Sobolev spaces

In this section, we list some definitions and propositions (some of them are new in our case, e.g. Proposition 2.3(4), Remark 2.2, Remark 2.4 for Theorem 2.10) about Musielak–Orlicz–Sobolev spaces for the readers to refer. Firstly, we give the definition of "N-function" and "generalized N-function" as follows.

Definition 2.1. A function $A : \mathbb{R} \to [0, +\infty)$ is called an N-function, denoted by $A \in N$, if A is even and convex, $A(0) = 0, 0 < A(t) \in C^0$ for $t \neq 0$, and the following conditions hold

$$\lim_{t\to 0+} \frac{A(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to +\infty} \frac{A(t)}{t} = +\infty.$$

A function $A : \Omega \times \mathbb{R} \to [0, +\infty)$ is called a generalized N-function, denoted by $A \in N(\Omega)$, if for each $t \in [0, +\infty)$, the function $A(\cdot, t)$ is measurable, and for a.e. $x \in \Omega$, we have $A(x, \cdot) \in N$.

Let $A \in N(\Omega)$, the Musielak–Orlicz space $L^A(\Omega)$ is defined by

$$L^A(\Omega) := \left\{ u : u \text{ is a measurable real function, and } \exists \lambda > 0 \text{ such that } \int_\Omega A\left(x, \frac{|u(x)|}{\lambda}\right) \mathrm{d}x < +\infty \right\}$$

with the (Luxemburg) norm

$$\|u\|_{L^A(\Omega)} = \|u\|_A := \inf\left\{ \lambda > 0 : \int_\Omega A\left(x, \frac{|u(x)|}{\lambda}\right) \mathrm{d}x \le 1 \right\}.$$

The Musielak–Sobolev space $W^{1,A}(\Omega)$ can be defined by

$$W^{1,A}(\Omega) := \{ u \in L^A(\Omega) : |\nabla u| \in L^A(\Omega) \}$$

with the norm

$$\|u\|_{W^{1,A}(\Omega)} = \|u\|_{1,A} := \|u\|_A + \|\nabla u\|_A,$$

where $\|\nabla u\|_A := \| \, |\nabla u| \, \|_A$.

Definition 2.2. We say that A satisfies Condition (Δ_2), if there exist a positive constant $K > 0$ and a nonnegative function $h \in L^1(\Omega)$ such that

$$A(x, 2t) \le K A(x, t) + h(x) \quad \text{for } x \in \Omega \text{ and } t \in \mathbb{R}.$$

A is called locally integrable if $A(\cdot, t_0) \in L^1(\Omega)$ for every $t_0 > 0$.

For $x \in \Omega$ and $t \ge 0$, we denote by $a(x, t)$ the right-hand derivative of $A(x, \cdot)$ at t, at the same time define $a(x, t) = -a(x, -t)$. Then $A(x, t) = \int_0^{|t|} a(x, s)\, \mathrm{d}s$ for $x \in \Omega$ and $t \in \mathbb{R}$.

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