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Asymptotic symmetries for fractional operators

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ABSTRACT

In this paper, we study equations driven by a non-local integrodifferential operator $\mathcal{L}_K$ with homogeneous Dirichlet boundary conditions. More precisely, we study the problem

$$\begin{cases} -\mathcal{L}_K u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $2 < p < 2_s^* = \frac{2N}{N-2s}$, Ω is an open bounded domain in $\mathbb{R}^N$ for $N \geqslant 2$ and V is a L^∞ potential such that $-\mathcal{L}_K + V$ is positive definite. As a particular case, we study the problem

$$\begin{cases} (-\Delta)^s u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $(-\Delta)^s$ denotes the fractional Laplacian (with $0 < s < 1$). We give assumptions on V, Ω and K such that ground state solutions (resp. least energy nodal solutions) respect the symmetries of some first (resp. second) eigenfunctions of $-\mathcal{L}_K + V$, at least for p close to 2. We study the uniqueness, up to a multiplicative factor, of those types of solutions. The results extend those obtained for the local case.

1. Introduction

Non-local operators arise naturally in many different topics in physics, engineering and even finance. For examples, they have applications in crystal dislocation, soft thin films, obstacle problems [1,2], continuum

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mechanics [3], chaotic dynamics of classical conservative systems [4] and graph theory [5]. In this paper, we shall consider the non-local counterpart of semi-linear elliptic equations of the type

$$\begin{cases} -\Delta u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded domain with Lipschitz boundary, $2 < p < 2^*$ is a subcritical exponent (where $2^* := 2N/(N-2)$ if $N \geqslant 3$, $2^* = +\infty$ if $N = 2$) and $V \in L^\infty$ is such that $-\Delta + V$ is positive definite. Precisely, we are predominantly interested in the qualitative behavior of solutions to

$$\begin{cases} (-\Delta)^s u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.2}$$

where $(-\Delta)^s$ denotes the fractional Laplacian (with $0 < s < 1$) and $2 < p < 2_s^* := \frac{2N}{N-2s}$. Let us recall that, up to a normalization factor, $(-\Delta)^s$ may be defined [6] as follows: for $x \in \mathbb{R}^N$,

$$(-\Delta)^s u(x) := -c_{N,s} \lim_{\varepsilon \to 0} \int_{\complement B(x,\varepsilon)} \frac{u(y) - u(x)}{|y-x|^{N+2s}} \,\mathrm{d}y = -\tfrac{1}{2} c_{N,s} \int_{\mathbb{R}^N} \frac{u(x+y) - 2u(x) + u(x-y)}{|y|^{N+2s}} \,\mathrm{d}y$$

where $c_{N,s} := s 2^{2s}\, \Gamma(\frac{N+2s}{2}) / \big(\pi^{N/2}\, \Gamma(1-s)\big)$ is a positive constant chosen [7] to be coherent with the Fourier definition of $(-\Delta)^s$. This problem is variational and a ground state (resp. a least energy nodal solution) can be defined from the associated Euler–Lagrange functional—see [8] (resp. Section 2) for more details. In this paper, we would like to study the symmetries of those two types of variational solutions. In fact, we consider a more general setting: we are dealing with ground state and least energy nodal solutions to the following equation:

$$\begin{cases} -\mathcal{L}_K u + V(x)u = |u|^{p-2}u, & \text{in } \Omega, \\ u = 0, & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.3}$$

where $\mathcal{L}_K$ is the non-local operator defined as follows

$$\mathcal{L}_K u(x) := \int_{\mathbb{R}^N} \big(u(x+y) - 2u(x) + u(x-y)\big) K(y) \,\mathrm{d}y.$$

We shall assume that $K : \mathbb{R}^N \setminus \{0\} \to (0, +\infty)$ is a function such that $mK \in L^1(\mathbb{R}^N)$ where $m(x) := \min\{|x|^2, 1\}$ and we require the existence of $\theta > 0$ and $s \in (0,1)$ such that $K(x) \geqslant \theta |x|^{-(N+2s)}$ for any $x \in \mathbb{R}^N \setminus \{0\}$. We also require that $K(x) = K(-x)$ for any $x \in \mathbb{R}^N \setminus \{0\}$. In particular, we can consider $K(x) = \frac{1}{2} c_{N,s} |x|^{-(N+2s)}$ so that $-\mathcal{L}_K$ is exactly the fractional Laplacian operator $(-\Delta)^s$ as defined in (1.1) and (1.3) boils down to (1.2).

Let us point out that, in the current literature, there are several notions of fractional Laplacian, all of which agree when the problems are set on the whole $\mathbb{R}^N$, but some of them disagree in a bounded domain. The values $(-\Delta)^s u(x)$ are, as we said, consistent with the Fourier definition of $(-\Delta)^s$, namely $\mathcal{F}^{-1}(|\xi|^{2s} \mathcal{F} u)$ and also agree with the local formulation due to Caffarelli–Silvestre [9],

$$(-\Delta)^s u(x) = -C \lim_{t \to 0} \Big(t^{1-2s} \frac{\partial U}{\partial t}(x,t) \Big),$$

where $U : \mathbb{R}^N \times (0, \infty) \to \mathbb{R}$ is the solution to $\operatorname{div}(t^{1-2s} \nabla U) = 0$ and $U(x,0) = u(x)$. The fractional Laplacian defined in this way is also called *integral*. In a bounded domain Ω, as in [10], we choose to operate with it on restrictions to Ω of functions defined on $\mathbb{R}^N$ which are equal to zero on $\complement\Omega$. A different operator $(-\Delta)^s_{\text{spec}}$ called *regional, local* or *spectral* fractional Laplacian, largely utilized in literature, can be defined as the power of the Laplace operator $-\Delta$ via the spectral decomposition theorem. Let $(\lambda_k)_{k \geqslant 1}$ and $(e_k)_{k \geqslant 1}$