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Existence and multiplicity of solutions for the Kirchhoff equations with asymptotically linear nonlinearities

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ABSTRACT

In this paper we study the nonlinear Kirchhoff equations on the whole space. We show the existence, non-existence, and multiplicity of solutions to this problem with asymptotically linear nonlinearities. This result can be regarded as an extension of the result in Li et al. (2012).

1. Introduction and main result

In this paper, we study the existence, non-existence, and multiplicity of solutions to the following Kirchhoff equation in $\mathbb{R}^N$ $(N \geq 3)$:

$$-\left(1 + b\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx\right)\Delta u + V(x)u = f(u) \quad \text{in } \mathbb{R}^N, \tag{$\mathcal{P}$}$$

where $b > 0$ is a parameter, $V(x)$ and $f(u)$ satisfy the following hypotheses:

(V1) $V(x) \in C(\mathbb{R}^N, \mathbb{R})$ and $V(x) \equiv V(|x|) \geq V_0 > 0$ for all $x \in \mathbb{R}^N$;
(V2) $\lim_{|x|\to\infty} V(x) = V(\infty) \in (0, +\infty)$;
(F1) $f \in C(\mathbb{R}, \mathbb{R})$ and $\lim_{s\to 0} \frac{f(s)}{s} = 0$;

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(F2) there exists $l \in (\Lambda, V(\infty))$ such that $\lim_{|s|\to\infty} \frac{f(s)}{s} = l$, where Λ is the infimum of the spectrum of the Schrödinger operator $-\Delta + V$, i.e.,

$$\Lambda = \inf_{u\in H^1(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx}{\int_{\mathbb{R}^N} u^2 dx};$$

(F3) $f(-s) = f(s)$ for all $s \in \mathbb{R}$.

Let us define the functional $\mathcal{I}_b(u) : H_r^1(\mathbb{R}^N) \to \mathbb{R}$ by

$$\mathcal{I}_b(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx + \frac{b}{4}\left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V(x)u^2)dx\right)^2 - \int_{\mathbb{R}^N} F(u)dx, \tag{1}$$

where $H_r^1(\mathbb{R}^N)$ denotes a radial function Sobolev space and $F(u) = \int_0^u f(t)dt$. Then, the critical points of $\mathcal{I}_b \in C^1$ provide the solutions of $(\mathcal{P})$.

The problem $(\mathcal{P})$ is related to the stationary analogue of the Kirchhoff equation

$$u_{tt} - \left(1 + b\int_\Omega |\nabla_x u|^2 dx\right)\Delta_x u = g(x, u) \tag{2}$$

which was proposed by G. Kirchhoff in 1883 (see [1]) as a generalization of the well-known d'Alembert's wave equation

$$\rho\frac{\partial^2 u}{\partial t^2} - \left(\frac{P_0}{h} + \frac{E}{2L}\int_0^L \left|\frac{\partial u}{\partial x}\right|^2 dx\right)\frac{\partial^2 u}{\partial x^2} = g(x, u)$$

for free vibrations of elastic strings. Kirchhoff's model takes into account the changes in length of the string produced by transverse vibrations. Here, L is the length of the string, h is the area of the cross section, E is the Young modulus of the material, ρ is the mass density and P_0 is the initial tension. The early classical studies of Kirchhoff equation were those of S. Bernstein [2] and S. I. Pohožaev [3]. However, Eq. (2) received great attention only after that J.-L. Lions [4] proposed an abstract framework for the problem.

Recently, using variational method, J. Jin and X. Wu [5] obtained the existence of infinitely many radial solutions for problem $(\mathcal{P})$ with $V(x) = 1$ in $\mathbb{R}^N$ using the Fountain theorem. Next, A. Azzollini et al. [6] get a multiplicity result concerning the critical points of a class of functionals involving local and nonlocal nonlinearities, then they apply their result to the nonlinear elliptic Kirchhoff equation $(\mathcal{P})$ in $\mathbb{R}^N$ assuming on the local nonlinearity has the general hypotheses introduced by H. Berestycki and P.-L. Lions [7]. In [8], A. Azzollini presents a very simple proof of the existence of at least one nontrivial solution for a Kirchhoff-type equation on $\mathbb{R}^N$, for $N > 3$. In particular, in the first part of the paper he is interested in studying the existence of a positive solution to the elliptic Kirchhoff equation under the effect of a nonlinearity satisfying the general Berestycki–Lions assumptions. In the second part, he looks for ground states using minimizing arguments on a suitable natural constraint. Very recently, Y. Li et al. [9] used A. Azzollini's idea to study the existence of at least one positive radial solution to the following nonlinear Kirchhoff-type equation:

$$\left(a + \lambda\int_{\mathbb{R}^N}|\nabla u|^2 + \lambda b\int_{\mathbb{R}^N} u^2\right)[-\Delta u + bu] = f(u) \tag{3}$$

in $\mathbb{R}^N$, where $N \geq 3$, a, b are positive constants, and $\lambda \geq 0$ is a parameter. The main result does not assume the usual compactness conditions. A cut-off functional associated with (3) is utilized to obtain bounded Palais–Smale sequences. Their result can be regarded as an extension of a classical result for the semilinear equation

$$-\Delta u + bu = f(u)$$