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Blow-up result in a Cauchy viscoelastic problem with strong damping and dispersive

Mohammad Kafini *, Muhammad I. Mustafa

Department of Mathematics and Statistics, KFUPM, Dhahran 31261, Saudi Arabia

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ABSTRACT

In this paper we consider a Cauchy problem for a nonlinear viscoelastic equation with strong damping and dispersive terms. Under certain conditions on the initial data and the relaxation function, we prove a finite-time blow-up result.

1. Introduction

In [1], Messaoudi considered the following initial–boundary value problem:

$$\begin{cases} u_{tt} - \Delta u + \displaystyle\int_0^t g(t-\tau)\Delta u(\tau)d\tau + u_t|u_t|^{m-1} = u|u|^{p-1}, \quad \text{in } \Omega \times (0,\infty) \\ u(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0 \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\partial\Omega$, $p > 1$, $m \geq 0$, and $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a positive nonincreasing function. He showed, under suitable conditions on g, that solutions with negative initial energy blow up in finite time if $p > m$ and continue to exist if $m \geq p$. This result has been later pushed, by the same author [2], to certain solutions with positive initial energy. A similar result has been also obtained by Wu [3] using a different method.

In the absence of the viscoelastic term ($g = 0$), the problem has been extensively studied and many results concerning global existence and nonexistence have been proved. For instance, for the equation

$$u_{tt} - \Delta u + au_t|u_t|^{m-1} = b|u|^{p-1}u, \quad \text{in } \Omega \times (0,\infty) \tag{1.2}$$

$m, p \geq 1$, it is well known that, for $a = 0$, the source term $bu|u|^{p-1}$, $(p > 1)$ causes finite time blow up of solutions with negative initial energy (see [4]). The interaction between the damping and the source terms was first considered by Levine [5,6] in the linear damping case ($m = 1$). He showed that solutions with negative initial energy blow up in finite time. Georgiev and Todorova [7] extended Levine's result to the nonlinear damping case ($m > 1$). In their work, the authors introduced a different method and showed that solutions with arbitrary negative energy continue to exist globally 'in time'

* Corresponding author.
E-mail addresses: mkafini@kfupm.edu.sa (M. Kafini), mmustafa@kfupm.edu.sa (M.I. Mustafa).

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if $m \geq p$ and blow up in finite time if $p > m$ and the initial energy is sufficiently negative. This last blow-up result has been extended to solutions with negative initial energy by Messaoudi [8] and others. For results of same nature, we refer the reader to Levine and Serrin [9], Vitillaro [10], and Messaoudi and Said-Houari [11].

Problems of the form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(x,s)ds - \gamma \Delta u_t = 0, \quad x \in \Omega,\ t \geq 0, \tag{1.3}$$

in bounded domains, were extensively studied by many authors. In [12], Cavalcanti et al. considered (1.3) and obtained a global existence for $\gamma \geq 0$ and uniform exponential decay for $\gamma > 0$. This work was extended by Messaoudi and Tatar [13] to a situation where a nonlinear source term is competing with the damping induced by $-\gamma \Delta u_t$ and the integral term. Then in the case of $\gamma = 0$, the same authors [14] showed that the damping induced by the viscoelastic term is enough to ensure global existence and uniform decay of solutions provided that the initial data are in some stable set by introducing a new functional and using the potential well method. Recently, Wu [15] improved [14] by considering the nonlinear equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-s)\Delta u(x,s)ds + |u_t|^{m-1} u_t = |u|^{p-1} u, \quad x \in \Omega,\ t \geq 0,$$

and a general decay result was obtained. In the presence of strong damping term Δu_t and dispersive term Δu_{tt}, Xu et al. [16] considered the initial–boundary value problem for the following viscoelastic wave equation:

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds - \Delta u_t - \Delta u_{tt} + u_t = |u|^{p-1} u, \quad x \in \Omega,\ t \geq 0.$$

By introducing a family of potential wells, the authors not only obtained the invariant sets, but also proved the existence and nonexistence of a global weak solution under some conditions with low initial energy. Furthermore, they established a blow-up result for certain solutions with arbitrary positive initial energy.

In the case of unbounded domains, for the problem (1.2) in $\mathbb{R}^n$, we mention, among others, the work of Levine, Serrin and Park [17], Todorova [18,19], Messaoudi [20], and Zhou [21].

In [22], the following Cauchy problem

$$\begin{aligned} &u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t = |u|^{p-1} u, \quad x \in \mathbb{R}^N,\ t > 0, \\ &u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), \quad x \in \mathbb{R}^N, \end{aligned} \tag{1.4}$$

where g, u_0, u_1 are specific functions, was discussed by Kafini and Messaoudi. Under suitable conditions on the initial data and the relaxation function, they proved a finite-time blow-up result. Actually they extended the result of Zhou Yong [21]. In [23], the same authors pushed that result to prove a blow-up to the coupled system

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds = f_1(u,v), & \text{in } \mathbb{R}^N \times (0,\infty) \\ v_{tt} - \Delta v + \int_0^t h(t-s)\Delta v(x,s)ds = f_2(u,v), & \text{in } \mathbb{R}^N \times (0,\infty) \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \mathbb{R}^N \\ v(x,0) = v_0(x), \qquad v_t(x,0) = v_1(x), & x \in \mathbb{R}^N. \end{cases}$$

Our aim is to extend the result of [16], established in bounded domains, to the problem in unbounded domains. Namely, we consider the following Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(x,s)ds + u_t - \Delta u_t - \Delta u_{tt} = |u|^{p-1} u, & x \in \mathbb{R}^n,\ t > 0, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

We aim to obtain blow up results for solutions with negative initial energy. To achieve this goal some conditions have to be imposed on the relaxation function g, the initial data u_0 and u_1 and for $p > 1$ as well.

2. Preliminaries

In this section we present some material needed in the proof of our result. First, we state, without a proof, a local existence result, which can be established by adopting the arguments of [12,24,25]. For this reason, let us assume that

(G) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable function such that

$$1 - \int_0^\infty g(s)ds = l > 0, \qquad g'(t) \leq 0, \quad t \geq 0. \tag{2.1}$$