Nonlinear Analysis: Real World Applications 20 (2014) 36–51

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Coexistence solutions and their stability of an unstirred chemostat model with toxins☆

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ARTICLE INFO

Article history:
Received 31 August 2013
Received in revised form 21 April 2014
Accepted 26 April 2014

Keywords:
Chemostat
Coexistence solutions
Stability
Fixed point index theory
Perturbation technique

ABSTRACT

We study a reaction–diffusion system of the unstirred chemostat model with toxins in the N-dimensional case. Firstly, some sufficient conditions for the existence of positive steady-state solutions are obtained by means of the fixed point index theory. Secondly, the local structure of positive steady-state solutions and their stability are discussed by the bifurcation theory. Lastly, the effect of the toxins is investigated by the perturbation technique. The results show that when the parameter β, which measures the effect of toxins, is large enough, this model has no coexistence solution provided that the maximal growth rate b of the species v is greater than $\frac{\hat{\sigma}_1}{1-k}$, and all positive solutions of this model are controlled by a limit system provided that b belongs to the interval $(\frac{\sigma_1}{1-k}, \frac{\hat{\sigma}_1}{1-k})$.

1. Introduction

The chemostat is an experimental device for the continuous culture of microorganisms. Experimental studies have shown bacterial contamination may occur in the process of the long-term continuous culture of microorganisms (see [1–4] etc.). Therefore, the study of mathematical models for the continuous culture of microorganisms in the chemostat with toxins has recently been a problem of considerable interest (see [1,2,4–10] and the references therein).

This paper deals with an unstirred chemostat model with toxins in the N-dimensional case:

$$\begin{aligned} S_t &= \Delta S - auf_1(S) - bvf_2(S), \quad x \in \Omega,\ t > 0, \\ u_t &= \Delta u + auf_1(S) - \beta pu, \quad x \in \Omega,\ t > 0, \\ v_t &= \Delta v + b(1-k)vf_2(S), \quad x \in \Omega,\ t > 0, \\ p_t &= \Delta p + bkvf_2(S), \quad x \in \Omega,\ t > 0 \end{aligned} \tag{1.1}$$

with boundary conditions and initial conditions

$$\begin{aligned} &\frac{\partial S}{\partial n} + \gamma(x)S = S^0(x), \qquad \frac{\partial u}{\partial n} + \gamma(x)u = 0, \quad x \in \partial\Omega,\ t > 0, \\ &\frac{\partial v}{\partial n} + \gamma(x)v = 0, \qquad \frac{\partial p}{\partial n} + \gamma(x)p = 0, \quad x \in \partial\Omega,\ t > 0, \end{aligned} \tag{1.2}$$

☆ The work is supported by the Natural Science Foundation of China (Nos. 11001160, and 11271236) and the Program of New Century Excellent Talents in University of Ministry of Education of China (NCET-12-0894) and the Fundamental Research Funds for the Central Universities (GK201303008).
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http://dx.doi.org/10.1016/j.nonrwa.2014.04.002
1468-1218/

$$\begin{aligned}&S(x,0)=S_0(x)\geq 0, \qquad u(x,0)=u_0(x)\geq 0, \not\equiv 0, \quad x\in\overline{\Omega},\\&v(x,0)=v_0(x)\geq 0,\not\equiv 0, \qquad p(x,0)=p_0(x)\geq 0,\not\equiv 0, \quad x\in\overline{\Omega}.\end{aligned} \tag{1.3}$$

Here Ω is a bounded region in $R^N (N \geq 1)$ with smooth boundary $\partial\Omega$, n is the outward unit normal vector on $\partial\Omega$. $S(x,t)$ is the concentration of the nutrient at time t, $u(x,t)$, $v(x,t)$ are the concentrations of two species and $p(x,t)$ is the concentration of the toxin. a, b are the maximal growth rates of two species, respectively. The response function $f_i(S)=\frac{S}{a_i+S}$, $i=1,2$, where $a_i>0$ is the Michaelis–Menten constant. β and k are positive constants, and k is the fraction of consumption devoted to the production of the toxin, hence $0<k<1$. Moreover, $S^0(x)\geq 0,\not\equiv 0 (\forall x\in\partial\Omega)$ is the input concentration of the nutrient and is consistent with the initial condition $S_0(x)$ for $x\in\partial\Omega$. $\gamma(x)$ is continuous and non-negative on $\partial\Omega$.

The model is essentially the chemostat model studied by Hsu and Waltman [1]. However, we use the unstirred chemostat instead of the chemostat following Hsu and Waltman [11], Wu [12], Nie and Wu [5,6], Hsu et al. [13], etc. In [1], the authors gave a complete characterization of the competitive outcome of the well stirred chemostat model with toxins in terms of the relevant parameters in hyperbolic cases, including the globally asymptotical stability of the trivial solution and the two semi-trivial solutions and the instability of the unique positive solution. Zhu et al. studied the well stirred chemostat model with general yield functions of competition and toxins (see [10]). They attained the similar global stability and instability of nonnegative solutions as in [1], and the existence of stable limit cycles by the Hopf bifurcation theory as well, which illustrated the nonlinear oscillatory behavior occurring in the process of the continuous culture of microorganisms. However, the asymptotic behavior of the chemostat model is not clear in combination with the effects of diffusion and toxins.

The focus of this paper is to consider the asymptotic behavior of the above chemostat model in combination with the effects of diffusion and toxins. To figure out the asymptotic behavior of the system (1.1)–(1.3), we first study the corresponding steady-state system of (1.1)–(1.3):

$$\begin{aligned}&\Delta S - auf_1(S) - bvf_2(S) = 0, \quad x\in\Omega,\\&\Delta u + auf_1(S) - \beta pu = 0, \quad x\in\Omega,\\&\Delta v + b(1-k)vf_2(S) = 0, \quad x\in\Omega,\\&\Delta p + bkvf_2(S) = 0, \quad x\in\Omega\end{aligned} \tag{1.4}$$

with boundary conditions

$$\begin{aligned}&\frac{\partial S}{\partial n}+\gamma(x)S=S^0(x), \qquad \frac{\partial u}{\partial n}+\gamma(x)u=0, \quad x\in\partial\Omega,\\&\frac{\partial v}{\partial n}+\gamma(x)v=0, \qquad \frac{\partial p}{\partial n}+\gamma(x)p=0, \quad x\in\partial\Omega.\end{aligned} \tag{1.5}$$

Since only nonnegative solutions $S(x)$, $u(x)$, $v(x)$ and $p(x)$ are meaningful, we redefine $\hat{f}_i(S)$ for $S<0$ as follows:

$$\hat{f}_i(S)=\begin{cases}f_i(S), & S\geq 0,\\ S/a_i, & S<0.\end{cases}$$

It is easy to see $\hat{f}_i(S)\in C^1(-\infty,+\infty)$. We denote $\hat{f}_i(S)$ by $f_i(S)$ for simplicity.

The aim of this paper is to investigate the properties of coexistence solutions and their stability of the steady-state system (1.4)–(1.5). First of all, let $Q(x)=(1-k)p-kv$. Then

$$\Delta Q=0,\ x\in\Omega, \quad \frac{\partial Q}{\partial n}+\gamma(x)Q=0, \quad x\in\partial\Omega,$$

which implies $Q\equiv 0$ on $\overline{\Omega}$. Namely, $p=\alpha v$ with $\alpha=\frac{k}{1-k}$. Thus, (1.4)–(1.5) can be reduced into the following system

$$\begin{aligned}&\Delta S - auf_1(S) - bvf_2(S) = 0, \quad x\in\Omega,\\&\Delta u + auf_1(S) - \alpha\beta uv = 0, \quad x\in\Omega,\\&\Delta v + b(1-k)vf_2(S) = 0, \quad x\in\Omega,\\&\frac{\partial S}{\partial n}+\gamma(x)S=S^0(x), \qquad \frac{\partial u}{\partial n}+\gamma(x)u=0, \qquad \frac{\partial v}{\partial n}+\gamma(x)v=0, \quad x\in\partial\Omega.\end{aligned} \tag{1.6}$$

Hence, we only need to study the properties of coexistence solutions and their stability of the reduced system (1.6).

The contents of the paper are as follows: In Section 2, we introduce some well-known lemmas related to our study as preliminaries. The goal of Section 3 is to establish some sufficient and necessary conditions for the existence of positive solutions to (1.6). Moreover, the stability of trivial and semi-trivial solutions is also obtained. In Section 4, we investigate the bifurcation structure and stability of positive solutions to (1.6) by the bifurcation theory. Finally, in Section 5, the effects of the toxins are studied. Some stability and asymptotic behavior of positive solutions to (1.6) are gained.