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An existence result for a mixed variational problem arising from Contact Mechanics

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ABSTRACT

We consider a mixed variational problem involving a nonlinear, hemicontinuous, generalized monotone operator. The proposed problem consists of a variational equation in a real reflexive Banach space and a variational inequality in a subset of a second real reflexive Banach space. We investigate the existence of the solution using a fixed point theorem for set valued mapping. An example arising from Contact Mechanics illustrates the theory.

1. Introduction

The present paper focuses on the following mixed variational problem.

Problem 1. Given $f \in X'$, find $(u, \lambda) \in X \times \Lambda$ so that

$$(Au, v)_{X',X} + b(v, \lambda) = (f, v)_{X',X} \quad \text{for all } v \in X, \tag{1}$$

$$b(u, \mu - \lambda) \le 0 \quad \text{for all } \mu \in \Lambda. \tag{2}$$

Here and everywhere below X' denotes the dual of the space X and Λ is a subset of a space Y.

If X and Y are Hilbert spaces and $A : X \to X$ is a symmetric, continuous and strongly monotone operator, then we can write the following *saddle point problem*:

$$a(u, v) + b(v, \lambda) = (\tilde{f}, v)_X \quad \text{for all } v \in X, \tag{3}$$

$$b(u, \mu - \lambda) \le 0 \quad \text{for all } \mu \in \Lambda; \tag{4}$$

herein $a : X \times X \to \mathbb{R}$ is the bilinear, symmetric, continuous, X-elliptic form $a(u, v) = (Au, v)_X$ and $\tilde{f}$ is the unique element of X so that $(f, v)_{X',X} = (\tilde{f}, v)_X$ for all $v \in X$. If, in addition, $b(\cdot, \cdot) : X \times Y$ is a bilinear continuous form satisfying the "inf–sup property"

$$\exists\, \alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \frac{b(v, \mu)}{\|v\|_X \|\mu\|_Y} \ge \alpha$$

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and Λ is a closed convex subset of Y so that $0_Y \in \Lambda$, then the problem (3)–(4) has a unique solution $(u, \lambda) \in X \times \Lambda$ which is the unique saddle point of the following functional

$$\mathcal{L} : X \times \Lambda \to \mathbb{R} \quad \mathcal{L}(v, \mu) = \frac{1}{2}a(v, v) - (\tilde{f}, v)_X + b(v, \mu), \tag{5}$$

see e.g. [1,2]. The saddle point problem (3)–(4) can be related to the weak formulation of a class of unilateral frictionless or bilateral frictional contact problems, for linearly elastic materials, see for instance [2,3]. For a class of generalized saddle point problems related to the weak solvability of contact models involving a particular class of nonlinearly elastic materials we refer the reader to [4,5]; the weak solution of such a generalized saddle point problem is the unique fixed point of a single valued operator which is defined by means of the unique solution of an intermediate saddle point problem.

The current work focuses on a new theoretical result which will allow to explore contact models for another class of nonlinearly elastic materials; the key herein is not the saddle point theory; the key here is a fixed point theorem for set valued mapping. It is worth mentioning that mixed weak formulations in Contact Mechanics are appropriate approaches to efficiently approximate the weak solutions; see e.g. [6,7,3,8] for modern numerical techniques. The study on this direction is in progress. For a more complex view on mixed variational formulations in Mechanics we refer the reader also to [9–14].

In the present paper we shall study Problem 1 under the following assumptions.

Assumption 1. $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ are two real reflexive Banach spaces.

Assumption 2. Λ is a closed convex bounded subset of Y so that $0_Y \in \Lambda$.

Assumption 3. There exists a functional $h : X \to \mathbb{R}$ so that:

- (i_1) $h(tw) = t^r h(w)$ for all $t > 0$, $w \in X$ and $r > 1$;
- (i_2) $(Av - Au, v - u)_{X',X} \geq h(v - u)$ for all $u, v \in X$;
- (i_3) If $(x_n)_n \subset X$ is a sequence so that $x_n \rightharpoonup x$ in X as $n \to \infty$, then $h(x) \leq \limsup_{n\to\infty} h(x_n)$.

Notice that (i_1) and (i_2) in Assumption 3 express a generalized monotonicity property for the operator $A : X \to X'$. According to the literature, the operator A is a relaxed h-monotone operator, see for example [15]; see also [16–20] for various generalizations of monotonicity such as pseudomonotonicity, quasimonotonicity, semimonotonicity, relaxed monotonicity.

Assumption 4. The operator $A : X \to X'$ is hemicontinuous, i.e., for all $u, v \in X$, the mapping $f : \mathbb{R} \to (-\infty, +\infty)$, $f(t) = (A(u + tv), v)_{X',X}$ is continuous at 0.

Assumption 5. $\frac{(Au,u)_{X',X}}{\|u\|_X} \to \infty$ as $\|u\|_X \to \infty$.

Assumption 6. The form $b : X \times Y \to \mathbb{R}$ is bilinear. In addition,

- for each sequence $(u_n)_n \subset X$ so that $u_n \rightharpoonup u$ in X as $n \to \infty$ we have $b(u_n, \mu) \to b(u, \mu)$ as $n \to \infty$, for all $\mu \in \Lambda$.
- for each sequence $(\lambda_n)_n \subset Y$ so that $\lambda_n \rightharpoonup \lambda$ in Y as $n \to \infty$, we have $b(v, \lambda_n) \to b(v, \lambda)$ as $n \to \infty$, for all $v \in X$.

In the present paper we shall prove that, under Assumptions 1–6, Problem 1 has at least one solution. Assumptions 1–6 impose a new technique in order to handle Problem 1, namely a fixed point technique involving a set valued mapping, instead of a saddle point technique. Let us recall here the main tool we use.

Theorem 1. *Let $\mathcal{K} \neq \emptyset$ be a convex subset of a Hausdorff topological vector space $\mathcal{E}$. Let $F : \mathcal{K} \to 2^{\mathcal{K}}$ be a set valued map so that*

(h_1) *for each $u \in \mathcal{K}$, $F(u)$ is a nonempty convex subset of $\mathcal{K}$;*
(h_2) *for each $v \in \mathcal{K}$, $F^{-1}(v) = \{u \in \mathcal{K} : v \in F(u)\}$ contains an open set O_v which may be empty;*
(h_3) $\bigcup_{v\in\mathcal{K}} \mathcal{O}_v = \mathcal{K}$;
(h_4) *there exists a nonempty set $\mathcal{V}_0$ contained in a compact convex subset $\mathcal{V}_1$ of $\mathcal{K}$ so that $\mathcal{D} = \bigcap_{v\in\mathcal{V}_0} \mathcal{O}_v^c$ is either empty or compact.*

Then, there exists $u_0 \in \mathcal{K}$ so that $u_0 \in F(u_0)$.

We note that $2^{\mathcal{K}}$ denotes the family of all subsets of $\mathcal{K}$, and $\mathcal{O}_v^c$ is the complement of $\mathcal{O}_v$ in $\mathcal{K}$. For a proof of this theorem we refer to [21].

We end this introductive part by specifying the structure of the rest of the paper. In Section 2 an existence result for an intermediate problem is given. In Section 3 we use the intermediate result to prove that, under Assumptions 1–6, Problem 1 has at least one solution. In Section 4 we give an example of functional spaces X and Y, operator A, bilinear form $b(\cdot,\cdot)$ and subset Λ so that Assumptions 1–6 are fulfilled. In the last section we discuss a contact model related to the example given in Section 4.