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Regularity results and exponential growth for pullback attractors of a non-autonomous reaction–diffusion model with dynamical boundary conditions

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ABSTRACT

In this paper, we prove some regularity results for pullback attractors of a non-autonomous reaction–diffusion model with dynamical boundary conditions considered in Anguiano (2011). Under certain assumptions of the nonlinear terms we show a regularity result for the unique solution of the problem. We establish a general result about boundedness of invariant sets for the associated evolution process in the norm of the domain of the spatial linear operator appearing in the equation. As a consequence, we deduce that the pullback attractors of the model are bounded in this domain norm. After that, under additional assumptions, some exponential growth results for pullback attractors when time goes to $-\infty$ are proved.

1. Introduction and setting of the problem

PDE problems with dynamic boundary conditions have the main characteristic of involving the time derivative of the unknown on the boundary of the domain. Although not so often considered in the literature as other boundary conditions (Dirichlet, Neumann, or Robin), dynamic boundary conditions also appear in a wide variety of applied problems.

Its use dates back at least to 1901 in the context of heat transfer in a solid in contact with a moving fluid (see the pioneering work by Peddie [1] in 1901, by March and Weaver [2] in 1928, by Peek [3] in 1929, by Langer [4] in 1932, and by Bauer [5] in 1953).

From the second half of the 20th century until today, they have been studied in many disciplines. Continuing with the topic of heat transfer between a domain and its boundary, see the more recent Refs. [6,7] and other variants, like heat transfer in two phase medium (e.g., cf. [8–10]; see [11] for a detailed study of a Cahn–Hilliard problem, a natural higher order generalization of the reaction–diffusion equation); problems in fluid dynamics (cf. [12] among others); diffusion phenomena (see [13]), in particular, diffusion in porous medium (e.g., cf. [14], beside the already cited Ref. [3]); probability theory and mathematical modeling in Biology (cf. [15]); thermoelasticity (cf. [16]); thermal energy storage devices (cf. [17]); chemical engineering (cf. [18]); semiconductor devices (cf. [19]), etc. (see [20] for more problems and classical references). A detailed physical interpretation of this boundary condition can be found in [21]. For a more abstract point of view the reader is referred to [22–24] among others.

The study of evolution equations with dynamic boundary conditions from the mathematical point of view dates back to 1961, when J.L. Lions [25] treated such equations and gave weak solutions by means of the variational method. Since then,

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1 Prof. José Real deceased on 2012. M.A. and P.M.-R. would like to dedicate this paper to his memory.

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this issue has been investigated to a large extent. In particular, the reaction–diffusion equation with dynamical boundary conditions arises in hydrodynamics and the heat transfer theory (see the above references), and this problem has strong background in mathematical physics. While some of the initial studies were devoted to existence results (e.g., cf. [26,22,27,23,28]), after that, the *borderline* (critical exponents) where blow-up phenomena appears or well-posedness holds was analyzed (e.g., cf. [24]), and some others papers are focused directly on well-posedness and the long-time behavior under suitable dissipativity assumptions, including equilibria, description of convergence, different type of attractors, attracting w.r.t. several metrics, etc. (cf. [29–31,10,32–34]).

By last in this brief introduction, let us recall that regularity issues concerning problems with dynamical boundary conditions have also been intensively analyzed. This is due to several facts. For instance, regularity properties allow to manage in different ways proofs of existence with different techniques, not only variational, but also by fixed points, using density arguments combined with approximating problems. Regularity features are also useful to deal with different flows in several phase–spaces, to improve attraction properties, and to implement more sophisticated numeric schemes. For example, regularity properties of several problems with dynamical boundary conditions, even of higher order, as wave equations and Cahn–Hilliard equations, with dynamic boundary conditions, have been addressed in [35–37].

Let us introduce the model we will be involved with in this paper. Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth enough boundary.

We consider the non-autonomous reaction–diffusion equation

$$\frac{\partial u}{\partial t} - \Delta u + \kappa u + f(u) = h(t) \quad \text{in } \Omega \times (\tau, \infty), \tag{1}$$

with the dynamical boundary condition

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial \vec{n}} + g(u) = \rho(t) \quad \text{on } \partial\Omega \times (\tau, \infty), \tag{2}$$

where $\vec{n}$ is the outer normal to $\partial\Omega$, and the initial conditions

$$u(x, \tau) = u_\tau(x) \quad \text{for } x \in \Omega, \tag{3}$$

$$u(x, \tau) = \psi_\tau(x) \quad \text{for } x \in \partial\Omega, \tag{4}$$

where $\tau \in \mathbb{R}$ is an initial time, and $\kappa > 0$, $u_\tau \in L^2(\Omega)$, $\psi_\tau \in L^2(\partial\Omega)$, $h \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$, and $\rho \in L^2_{\text{loc}}(\mathbb{R}; L^2(\partial\Omega))$ are given.

We also assume that the functions f and $g \in C(\mathbb{R})$ are given, and satisfy that there exist constants $p \geq 2$, $q \geq 2$, $\alpha_1 > 0$, $\alpha_2 > 0$, $\beta > 0$, and $l > 0$, such that

$$\alpha_1 |s|^p - \beta \leq f(s)s \leq \alpha_2 |s|^p + \beta \quad \forall s \in \mathbb{R},$$

$$\alpha_1 |s|^q - \beta \leq g(s)s \leq \alpha_2 |s|^q + \beta \quad \forall s \in \mathbb{R},$$

$$(f(s) - f(r))(s - r) \geq -l(s - r)^2 \quad \text{and} \quad (g(s) - g(r))(s - r) \geq -l(s - r)^2 \quad \forall s,\ r \in \mathbb{R}.$$

It is easy to deduce from above that there exists a constant $C > 0$ such that

$$|f(s)| \leq C(1 + |s|^{p-1}), \qquad |g(s)| \leq C(1 + |s|^{q-1})$$

for all $s \in \mathbb{R}$.

Let us denote

$$\mathcal{F}(s) := \int_0^s f(r)dr \quad \text{and} \quad \mathcal{G}(s) := \int_0^s g(r)dr.$$

Then, there exist positive constants $\widetilde{\alpha}_1$, $\widetilde{\alpha}_2$, and $\widetilde{\beta}$ such that

$$\widetilde{\alpha}_1 |s|^p - \widetilde{\beta} \leq \mathcal{F}(s) \leq \widetilde{\alpha}_2 |s|^p + \widetilde{\beta} \quad \forall s \in \mathbb{R}, \tag{5}$$

and

$$\widetilde{\alpha}_1 |s|^q - \widetilde{\beta} \leq \mathcal{G}(s) \leq \widetilde{\alpha}_2 |s|^q + \widetilde{\beta} \quad \forall s \in \mathbb{R}. \tag{6}$$

As we mentioned before, this model arises in different areas, especially in population growth, chemical reactions and heat conduction. For instance, in the case of a heat transfer in a medium Ω, Eq. (1) is a heat equation including an internal heat source. On the other hand, the heat flow from inside Ω to the boundary is $-\frac{\partial u}{\partial \vec{n}}$. The accumulation rate of heat on the boundary is $\frac{\partial u}{\partial t}$ so that, if we consider a source term on $\partial\Omega$, we have the dynamical boundary condition (2) (see the works by Constantin and Escher [38] and by Goldstein [21] for more details). Since the source terms may represent engineering or physic devices during the procedure (or noise perturbations, as in [39]), it is natural that they are time-depending, to indicate the time where they are acting with different intensities. This makes sense to consider some of the approaches of non-autonomous dynamical systems, as will be recalled below.

Before to continue with the setting of the problem, let us introduce some notation that will be useful in the sequel.