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Zero viscosity and diffusion vanishing limit of the incompressible magnetohydrodynamic system with perfectly conducting wall

Zhonglin Wu [a,b,*], Shu Wang [a]

[a] College of Applied Science, Beijing University of Technology, Ping Le Yuan 100, Beijing 100124, PR China

[b] Department of Mathematics, Huanghuai University, Zhumadian 463000, PR China

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ABSTRACT

This work investigates the incompressible viscous and diffusive MHD system with perfectly conducting wall boundary condition. For different horizontal and vertical viscosities and magnetic diffusions, when they go to zero with different or same speeds, we establish the convergence to the ideal inviscid MHD system and the anisotropic inviscid MHD system by constructing the exact boundary layer functions.

1. Introduction

In this paper we consider zero viscosity and diffusion vanishing limit for the three-dimensional incompressible viscous and diffusive magnetohydrodynamic (MHD) system with Dirichlet boundary condition for the velocity and perfectly conducting wall boundary condition for the magnetic field

$$\partial_t u^{\eta,\nu} + u^{\eta,\nu}\cdot\nabla u^{\eta,\nu} - \nu_1\partial_z^2 u^{\eta,\nu} - \eta_1\Delta_{x,y}u^{\eta,\nu} + \nabla p^{\eta,\nu} = b^{\eta,\nu}\cdot\nabla b^{\eta,\nu}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.1}$$

$$\partial_t b^{\eta,\nu} + u^{\eta,\nu}\cdot\nabla b^{\eta,\nu} - \nu_2\partial_z^2 b^{\eta,\nu} - \eta_2\Delta_{x,y}b^{\eta,\nu} = b^{\eta,\nu}\cdot\nabla u^{\eta,\nu}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.2}$$

$$div u^{\eta,\nu} = 0, \qquad div b^{\eta,\nu} = 0, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.3}$$

$$u^{\eta,\nu} = 0, \qquad b^{\eta,\nu}\cdot n = 0, \qquad curl b^{\eta,\nu}\times n = 0, \quad (x,y,z,t)\in\partial\Omega\times(0,T) \tag{1.4}$$

$$u^{\eta,\nu}(t=0) = u_0^{\eta,\nu}, \qquad b^{\eta,\nu}(t=0) = b_0^{\eta,\nu}, \qquad div u_0^{\eta,\nu} = 0, \qquad div b_0^{\eta,\nu} = 0, \quad (x,y,z)\in\Omega. \tag{1.5}$$

Here $\Omega = \mathbb{R}^2\times[0,\infty)$, the constants η_1 or η_2 and ν_1 or ν_2 represent the horizontal and vertical viscosities or magnetic diffusions coefficient respectively, $u^{\eta,\nu}$, $p^{\eta,\nu}$ and $b^{\eta,\nu}$ are the velocity, the pressure and the magnetic field, $\Delta_{x,y}$ denotes the two-dimensional Laplace operator in the variables x and y, and n is the outward normal vector.

* Corresponding author at: College of Applied Science, Beijing University of Technology, Ping Le Yuan 100, Beijing 100124, PR China.
E-mail addresses: wuzhonglin03@126.com (Z. Wu), wangshu@bjut.edu.cn (S. Wang).

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The incompressible viscous and diffusive MHD system (1.1)–(1.3) in the whole space or with slip/no-slip boundary conditions has been studied extensively, so there is a lot of literature on the well-posedness, regularity and asymptotic limit topic, see [1–9] and references therein. Some authors have done a lot on some regularity criterions, see [2,3,5,7,8] and references therein. For example, when $\eta_1 > 0, \eta_2 > 0, \nu_1 > 0$ and $\nu_2 > 0$, the MHD system in the whole space and in the bounded domain with no-slip boundary condition for the velocity and with slip boundary condition for the magnetic field has a unique global classical solution for smooth initial data when space dimension $d = 2$, but when $d = 3$, there exists a global weak solution for a class of initial data, see [4,6]. Xiao, Xin and Wu investigate the solvability, regularity and vanishing viscosity limit of the incompressible viscous MHD with slip without friction boundary conditions, see [9]. Wang and Xin also consider boundary layers problem with no-slip boundary conditions and other boundary conditions for the magnetic field in some particular domains, they give two examples to show viscous boundary layers and their stability of the shear flow and a class of special steady states of ideal inviscid MHD system, see [10]. In [11], Masmoudi obtained viscosity vanishing limit for the Navier–Stokes equations with Dirichlet boundary condition by using energy method and constructing the exact boundary layer functions. Similar as the viscosity vanishing limit for the Navier–Stokes equations, we point out that the zero-viscosity and diffusion vanishing limit for incompressible viscous and diffusive MHD system with perfectly conducting wall at the boundary, is a interesting problem because of the formation of the boundary layer, see [12,2,13–18,11,19–22] and related references. So far, we do not find any zero viscosity vanishing limit results for MHD system with Dirichlet boundary condition for the velocity and perfectly conducting wall boundary condition for the magnetic field.

The above works motivate us to consider the asymptotic limit of (1.1)–(1.5) in the half space $\mathbb{R}^3_+$. We mention here that our problem is more complicated than problem considered by Masmoudi because of presence of the magnetic field. The main difficulty is due to the Dirichlet boundary conditions for the velocity and perfectly conducting wall for the magnetic field. First, let us have a look at the formal asymptotic limit for MHD system. Setting $\nu = (\nu_1, \nu_2) \to 0$ in (1.1)–(1.5), we have the following three-dimensional anisotropic MHD system formally

$$\partial_t u^{\eta,0} + u^{\eta,0}\cdot\nabla u^{\eta,0} - \eta_1\Delta_{x,y}u^{\eta,0} + \nabla p^{\eta,0} = b^{\eta,0}\cdot\nabla b^{\eta,0}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.6}$$

$$\partial_t b^{\eta,0} + u^{\eta,0}\cdot\nabla b^{\eta,0} - \eta_2\Delta_{x,y}b^{\eta,0} = b^{\eta,0}\cdot\nabla u^{\eta,0}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.7}$$

$$divu^{\eta,0} = 0, \qquad divb^{\eta,0} = 0, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.8}$$

$$u_3^{\eta,0} = 0, b_3^{\eta,0} = 0, \quad (x,y,z,t)\in\partial\Omega\times(0,T) \tag{1.9}$$

$$u^{\eta,0}(t=0) = u_0^{\eta,0}, \qquad b^{\eta,0}(t=0) = b_0^{\eta,0}, \qquad divu_0^{\eta,0} = 0, \qquad divb_0^{\eta,0} = 0, \quad (x,y,z)\in\Omega \tag{1.10}$$

Second, setting $\eta = (\eta_1, \eta_2) \to 0$ in (1.6)–(1.10), we obtain the following ideal inviscid incompressible MHD system

$$\partial_t u^{0,0} + u^{0,0}\cdot\nabla u^{0,0} + \nabla p^{0,0} = b^{0,0}\cdot\nabla b^{0,0}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.11}$$

$$\partial_t b^{0,0} + u^{0,0}\cdot\nabla b^{0,0} = b^{0,0}\cdot\nabla u^{0,0}, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.12}$$

$$divu^{0,0} = 0, \qquad divb^{0,0} = 0, \quad (x,y,z,t)\in\Omega\times(0,T) \tag{1.13}$$

$$u^{0,0}\cdot n = -u_3^0 = 0, \qquad b^{0,0}\cdot n = -b_3^{0,0} = 0, \quad (x,y,z,t)\in\partial\Omega\times(0,T) \tag{1.14}$$

$$u^{0,0}(t=0) = u_0^{0,0}, \qquad b^{0,0}(t=0) = b_0^{0,0}, \qquad divu_0^{0,0} = 0, \qquad divb_0^{0,0} = 0, \quad (x,y,z)\in\Omega. \tag{1.15}$$

The aim of this paper is to prove the above formal limits rigorously.

The paper is organized as follows. Section 2 gives the main results of this paper. The proofs of the viscosity and diffusion vanishing limit results for no-slip and perfectly conducting wall boundary case will be presented in Section 3.

2. The main results

In order to give our main theorems in this section, we first recall the following classical results on the existence of sufficiently regular solutions of the incompressible ideal MHD system (see [4,6]) and the incompressible anisotropic MHD system, which can be proved similarly as in the ideal MHD system, so we shall omit them.

Proposition 2.1. *Assume $(u_0^{0,0}, b_0^{0,0})$ satisfy $u_0^{0,0}, b_0^{0,0} \in H^s(\Omega)$, $s > \frac{3}{2}+1$, $divu_0^{0,0} = 0$, $divb_0^{0,0} = 0$. Then there exist $0 < T^* \le \infty$, the maximal existence time, and a unique smooth solution $(u^{0,0}, p^{0,0}, b^{0,0})$ defined on $[0, T^*)$ of (1.11)–(1.15) satisfying the following inequality:*

$$\sup_{0\le t\le T}(\|(u^{0,0}, b^{0,0})\|_{H^s(\Omega)} + \|(\partial_t u^{0,0}, \partial_t b^{0,0})\|_{H^{s-1}(\Omega)}) \le C(T), \quad \forall\, T < T^*.$$

Moreover, if $(b_0^{0,0}\cdot n)(x,y,z)|_{\partial\Omega} = 0$, $(curlb_0^{0,0}\times n)(x,y,z)|_{\partial\Omega} = 0$, then $(b^{0,0}\cdot n)(x,y,z,t)|_{\partial\Omega} = 0$, $(curlb^{0,0}\times n)(x,y,z,t)|_{\partial\Omega} = 0$ for all $0\le t\le T^*$. If so, we call the initial data to be well-prepared initial data.