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Subcritical bifurcation of free elastic shell of biological cluster

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ABSTRACT

In this paper we will investigate symmetry-breaking bifurcation of equilibrium forms of biological cluster. A biological cluster is a two-dimensional analogue of a gas balloon. The cluster boundary is connected with its kernel by elastic links. The inside part is filled with compressed gas or fluid. Equilibrium forms of biological cluster can be found as solutions of a certain second order ordinary functional-differential equation with four physical parameters: an elasticity coefficient $\alpha > 0$ of boundary, an elasticity coefficient $\beta > 0$ of links and two parameters $\eta, \nu > 0$ describing compressed gas or fluid. For each multiparameter $(\alpha, \beta, \eta, \nu)$ this equation possesses a radially symmetric solution. In Guze and Janczewska (2014) we proved the existence of symmetry-breaking bifurcation with respect to the ratio $\tau = \beta/\alpha$ of elasticity coefficients. Now our aim is to describe bifurcation branches. Namely, applying a finite-dimensional reduction and a key function method we will prove the subcritical behaviour of biological cluster.

1. Introduction

In this paper we will investigate bifurcation with loss of symmetry for a free elastic shell of biological cluster (see Fig. 1). We will continue our research that has been started in [1].

Our study was motivated by gas balloons. Namely, we are interested in the anatomy and behaviour of the part of a balloon that is called an envelope (see Fig. 2).

A biological cluster is a two-dimensional analogue of a gas balloon. It has an elastic boundary connected with its kernel by elastic links. The inside part of biological cluster is filled with compressed gas or fluid (see Fig. 1). This term was proposed by A. Borisovich and H. Treder in [2]. Furthermore, we can treat a parachute in a balloon envelope as an example of biological cluster, because its height is much smaller than the length of a top rim—a horizontal tape between the parachute and the rest of envelope. Then a shell of biological cluster is a top rim of parachute, a kernel of biological cluster corresponds to a circular deflation panel, and vertical tapes between panels of parachute are elastic links (compare Fig. 2).

The investigation of symmetry-breaking bifurcation in a two-dimensional model describing nonlinear deformations of a free elastic shell of biological cluster is very important, because it is the introduction to searching three-dimensional models, starting with a special case in which a side surface is formed by the shape of a horizontal crosscut, for example cylindrical balloons.

In [3–6] Avner Friedman and his coauthors have presented a technique for studying symmetry-breaking bifurcation for free boundary problems. We paid special attention to Friedman's joint work with A. Borisovich [3], where a certain elliptic free boundary problem in the plane was treated. They studied the existence of symmetry-breaking bifurcation from radially

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1468-1218/

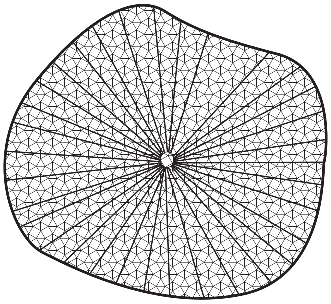

Fig. 1. A biological cluster.

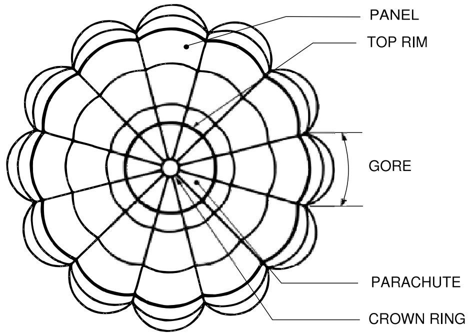

Fig. 2. A balloon envelope: view from above.

symmetric solutions by reducing the problem to one for which classical bifurcation methods may be applied. For classical methods of bifurcation theory we refer the reader to [7–11]. Their proof is based on the Crandall–Rabinowitz theorem on simple bifurcation points. Other examples of applications of the Crandall–Rabinowitz theorem to elasticity theory, biology and fluid mechanics can be found resp. in [12–18].

Let $C^m(2\pi)$, $m \in \mathbb{N} \cup \{0\}$, denote the Banach space of 2π-periodic C^m-smooth functions $r(\theta)$ with the standard norm

$$\|r\|_m = \sum_{k=0}^{m} \max_{\theta\in[0,2\pi]} |r^{(k)}(\theta)|, \tag{1}$$

where $r^{(k)}(\theta)$ denotes the kth derivative of $r(\theta)$ and $r^{(0)}(\theta) = r(\theta)$. It is well known that $C^m(2\pi)$ is continuously embedded into the Hilbert space $L^2(2\pi)$ with the scalar product

$$\left\langle f, g\right\rangle = \int_0^{2\pi} f(\theta)g(\theta)d\theta. \tag{2}$$

Equilibrium forms of biological cluster are described in polar coordinates by 2π-periodic C^{m+2}-smooth positive functions $r(\theta)$, $m \in \mathbb{N} \cup \{0\}$ (see Fig. 3).

They can be found as solutions of the following second order ordinary functional-differential equation

$$\alpha\frac{r^3(\theta) + 2r(\theta)r'^2(\theta) - r^2(\theta)r''(\theta)}{(r^2(\theta) + r'^2(\theta))^{3/2}} + \beta - \frac{\nu\eta}{S^{\nu+1}}r(\theta) = 0, \tag{3}$$

where $\alpha > 0$ is an elasticity coefficient of boundary, $\beta > 0$ is an elasticity coefficient of links, $\eta, \nu > 0$ are suitable physical parameters describing compressed gas or fluid inside the biological cluster, and S denotes the area of biological cluster, i.e.

$$S = S(r) = \frac{1}{2}\int_0^{2\pi} r^2(\theta)d\theta.$$

We are interested in radially symmetric solutions of (3). Substituting $r(\theta) \equiv r$ into (3) we get the algebraic equation

$$\alpha + \beta - \frac{\nu\eta}{\pi^{\nu+1}r^{2\nu+1}} = 0.$$

Finally, we get a solution given by

$$r_p = \left(\frac{\nu\eta}{\pi^{\nu+1}(\alpha+\beta)}\right)^{\frac{1}{2\nu+1}}$$

for all multiparameters $p = (\alpha, \beta, \eta, \nu) \in \mathbb{R}^4_+$.