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An exact solution for geophysical edge waves in the f-plane approximation

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ABSTRACT

We propose an exact implicit solution to the nonlinear geophysical edge-wave problem in the f-plane approximation. Adequate for this exact solution is the Lagrangian framework. We confirm that this solution is an example of a trapped wave.

1. Introduction

In geophysical fluid dynamics the Coriolis force due to Earth's rotation has an important effect. For waves propagating near the Equator, in order to get the governing equations much more tractable, the Coriolis force is approximated. A first approximation is the f-plane approximation, meaning a constant Coriolis parameter for which the latitudinal variations are ignored. This approximation applies reasonably well in small regions near the Equator where the curvature of the Earth is negligible (see the discussion in [1]). The β-plane approximation takes further into account that the Coriolis force may vary from point to point and introduces a linear variation with latitude of the Coriolis parameter. The β-plane approximation applies in regions which are within 5° latitude of the Equator (see the discussions in [2,3]).

We consider a non-zero sloping beach with the shoreline parallel to the Equator. Although difficult to visualize, among other waves which propagate on the surface of the water, there are the edge waves that progress along the shoreline and have the familiar characteristic of a trapped wave, that is, an amplitude which decays exponentially away from the shoreline. Mathematically, the edge waves are modelled by the Euler equations with appropriate kinematic and dynamic boundary conditions. Without taking into account the Coriolis force, the edge-wave solution to the full water-wave problem can be obtained by adapting the explicit deep water-wave solution discovered by Gerstner [4] (see [5,6] for modern expositions). An implicit form of the edge-wave solution was found in 1966 by Yih [7] (see also Mollo-Christensen [8]), while an explicit form was provided in 2001 by Constantin [9]. For those who may wish to explore the edge-wave phenomenon more deeply, we refer to the surveys [10–12], the papers [13,14]; a good source of discussions and references in relation to the physical importance of the edge-waves is [15].

Taking into account the Coriolis force at the level of f-plane approximation, Matioc [16] presented an explicit two-dimensional Gerstner type solution for geophysical deep-water waves. In the β-plane approximation, by using the insight provided by Gerstner's solution, Constantin [17] made significant modifications and found an explicit three-dimensional solution for equatorially trapped waves propagating eastward in a stratified inviscid fluid.

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The edge-wave solution given by Constantin [9] was extended by Matioc [18] to a setting with Coriolis effects: an explicit solution describing geophysical edge waves along a sloping beach with the shoreline parallel to the Equator in the f-plane approximation.

In this paper, we provide implicit Gerstner's type solution to the nonlinear geophysical edge-wave problem in the f-plane approximation. This solution describes in the Lagrangian framework geophysical edge waves propagating westwards or eastwards over a sloping beach with the shoreline parallel to the Equator. The dispersion relation obtained is compared to the one in the absence of Coriolis effects. We also confirm that this solution is an example of a trapped wave—the amplitude of the waves decays exponentially away from the shoreline.

2. The geophysical edge-wave problem in the *f*-plane approximation

We consider the Earth to be a perfect sphere of radius 6371 km which rotates around the north–south axis with a constant rotational speed $\omega = 73 \times 10^{-6}$ rad/s. In a rotating framework having the origin on the Earth's surface at a point with latitude ϕ, the x-axis horizontally due east, the y-axis horizontally due north and the z-axis upward (see Fig. 1), the governing equations for geophysical water waves are the Euler equations [3]

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} + 2(\mathbf{\Omega} \times \mathbf{u}) = -\frac{\nabla p}{\rho_0} + \mathbf{g}, \tag{1}$$

and the incompressibility condition

$$\nabla \cdot \mathbf{u} = 0. \tag{2}$$

In the above equations t is the time, $\mathbf{u}$ is the fluid velocity, $\mathbf{\Omega}$ is the rotation vector of Earth around the north–south axis, p is the pressure, ρ_0 is the constant density of the water and $\mathbf{g}$ is the gravity vector. In the rotating framework, the vector $\mathbf{\Omega}$ has the components

$$\mathbf{\Omega} = (0, \omega \cos\phi, \omega \sin\phi), \tag{3}$$

and thus, the Coriolis force becomes

$$2(\mathbf{\Omega} \times \mathbf{u}) = 2\omega(w\cos\phi - v\sin\phi, u\sin\phi, -u\cos\phi). \tag{4}$$

For a small fluid layer localized near the Equator, the rotation vector $\mathbf{\Omega}$ can be approximated by

$$\mathbf{\Omega} \approx (0, \omega, 0), \tag{5}$$

that is, $\mathbf{\Omega}$ is in the y-axis direction, the Earth sphericity plays no dynamic role. This approximation, termed the equatorial f-plane approximation, is physically realistic for certain equatorial flows, see the discussion in [1].

We consider a sloping beach with a shoreline parallel to the Equator and which forms an angle $\alpha \in \left(0, \frac{\pi}{2}\right)$ with the undisturbed water surface. Thus, x is the longshore coordinate and we introduce new coordinates y' and z', such that the xy'-plane is parallel to the sloping bed and the z'-axis normal to it (see Fig. 2).

In the f-plane approximation, the rotation vector $\mathbf{\Omega}$ has in the coordinate system $Oxy'z'$ the following components

$$\mathbf{\Omega} = (0, \omega\cos\alpha, -\omega\sin\alpha). \tag{6}$$

Therefore, in this case, the Euler equations of motion (1) take in the coordinate system $Oxy'z'$ the form

$$\begin{aligned}
&u_t + uu_x + vu_{y'} + wu_{z'} + 2\omega(w\cos\alpha + v\sin\alpha) = -\frac{p_x}{\rho_0}\\
&v_t + uv_x + vv_{y'} + wv_{z'} - 2\omega u\sin\alpha = -\frac{p_{y'}}{\rho_0} - g\sin\alpha\\
&w_t + uw_x + vw_{y'} + ww_{z'} - 2\omega u\cos\alpha = -\frac{p_{z'}}{\rho_0} - g\cos\alpha,
\end{aligned} \tag{7}$$

where u, v, w are the components of the fluid velocity $\mathbf{u}$ in this system and the gravity vector $\mathbf{g} = (0, -g\sin\alpha, -g\cos\alpha)$, $g = 9.8$ ms^{-2} being the gravitational acceleration.

The boundary conditions that together with Eqs. (7) and (2) define the geophysical edge-wave problem are the kinematic boundary conditions (KBC) as well as the dynamic boundary condition (DBC). The kinematic boundary conditions express the fact that the same particles always form the free water surface and that the sloping bed is impermeable. The dynamic boundary condition expresses the fact that on the free surface the pressure is equal to the constant atmospheric pressure denoted as p_0 (see the discussion in [19]).

3. A geophysical edge wave in the *f*-plane approximation

In this section we present an exact Gerstner-like solution of the geophysical edge-wave problem in the f-plane approximation. Adequate for this exact solution is the Lagrangian framework. In this framework, the coordinates of the fluid particles