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Mathematical and numerical approaches to a one-dimensional dynamic thermoviscoelastic contact problem

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ABSTRACT

In this paper, we propose numerical schemes for solving a nonlinear system which consists of a coupled partial differential equations and two conditions, called normal compliance contact condition and Barber's heat exchange condition. The convergence of numerical trajectories is shown by using a time discretization and passing the limit of the time step size. The uniqueness of the weak solution is proved as well. We derive the extensive form of an energy balance which will be a criterion to examine numerical stability. An example is provided to present and discuss numerical results.

1. Introduction

This one-dimensional *dynamic* contact problem is basically formulated by a coupled partial differential equations (PDEs) of linear thermoelasticity: an equation of motion and an equation of heat conduction, as was derived in Carlson's paper [1]. In general, this complicated nonlinear system may be considered as a simple example of thermodynamics. In everyday life, its applications can be found easily: for example, heating and cooling houses, cooking, regulating and mainlining our body temperature and so on. The recent research on thermodynamics can provide a wide range of applications in applied sciences such as biology, biochemistry and automobile engineering. However, all laws of thermodynamics are not applicable to atom or molecular levels and thus it does not explain the rate of chemical reaction.

The book [2, section 4.1] presents the general formalism of thermodynamical systems, and derives an equation which is thermodynamically consistent, by applying the second law of thermodynamics. The formalism can be applied to the PDEs to see their thermodynamic consistency. However, there are still some mathematical issues (see [2, section 4.5]), when we consider thermodynamics with side effects such as friction or wear. Concerning the thermodynamic consistency, readers may find more references (e.g., [3]).

Indeed, William Day (see his book [4]) introduces more advanced mathematical treatment for the one-dimensional PDEs with linear thermoelasticity. Unlike the standard boundary conditions considered in Day's book, we impose more applicable boundary conditions at two ends of the rod to complete our contact model. As for the usual contact problems of beams, the homogeneous essential boundary conditions are used at the left end. In addition to them, there are two natural boundary conditions at the right end which arise in our physical situation: one condition is *normal compliance* used in many papers (e.g., [5–8] and references therein) and another is Barber's heat exchange condition (see the papers [9,10]). Similar types of *quasistatic* contact problems were previously studied in many papers (e.g., [11–15]), applying the Signorini's contact

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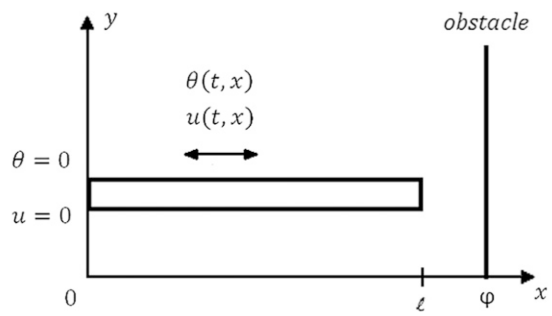

Fig. 2.1. Dynamic contact of a thermoviscoelastic rod with a deformable obstacle.

conditions instead of normal compliance. Those papers except [12] use different but unrealistic heat exchange conditions which are simpler than Barber's condition. Especially, in [15] a numerical scheme is proposed based on the Crank–Nicolson method and its convergence is proved as well.

In this work, we attach the viscous term to reformulate the equation of motion, due to the fact that proving the existence of solutions for the elastic case does not seem to be successful. From an engineering point of view, the viscosity is neither realistic nor important for most of engineering materials. Therefore, we will perform numerical experiments with a small quantity to approximate the purely thermoelastic case. Another mathematical difficulty to show the existence of solutions is to handle the heat exchange coefficient function in the radiation condition. Barber's three thermal contact conditions can be derived, by approximating a set-valued mapping. See the paper [12] for this issue. Even though the coefficient function is an unusual function, the existence of solutions for the quasistatic case can be proved in the paper [12], considering it as a measurable function. However, in the dynamic case, nicer assumptions on the coefficient function are required to show the existence results. If we apply Signorini contact conditions instead of normal compliance in this thermodynamic case, we have to impose auxiliary conditions into the input data of the coefficient function.

Although the two PDEs system and physical situation seem to be more complicated than other contact problems, proving the convergence of numerical trajectories is less technically complicated. Indeed, in order to show the boundedness of contact forces in dynamic contact problems with Signorini contact conditions, we need to verify an additional condition, called the strong pointedness.

This contact problem will play a fundamental role in being able to extend into more realistic contact models such as switches or actuators in Micro-electro-mechanical systems (MEMS). The V-shape electro-thermal actuator which is motivated by MEMS system is carried out in the paper [16].

In order to compute numerical approximations, we use the combination of a time discretization on the time interval and the Galerkin approximation over the spatial domain. This fully discrete numerical scheme leads us to establish the recursive formula where we can obtain the next time step numerical solutions per each time step.

This paper is organized as follows. Section 2 introduces continuous formulations of our contact model. In Section 3, mathematical notations and background are illustrated. In Section 4, we set up a weak formulation which is equivalent to the original PDEs system. The main results of the existence are presented, as well. In Section 5, we present a detailed description of numerical schemes and the convergence theory, applying a time discretization into the weak formulation. In Section 6, the new form of energy balance is derived and the uniqueness of the weak solution is proved. In Sections 7 and 8, the fully discrete numerical schemes are proposed, numerical experiments are performed, and numerical results are presented. Section 9 concludes this with several important remarks.

2. Contact model

Fig. 2.1 illustrates our *physical* situation in which a thermoviscoelastic rod fixed at the left wall stretches out, its right end touches the right wall, and shrinks back. The displacement and velocity of the rod are denoted by $u = u(t, x)$ and $v = u_t(t, x)$, respectively, and its temperature distribution is denoted by $\theta = \theta(t, x)$, where $x \in [0, l]$ with its initial length l and the time variable $t \in [0, T]$ with the final time $T > 0$. The two walls are assumed to be kept at different temperatures. The right wall is assumed to be a stationary deformable obstacle located at $x = \varphi > 0$. The rod is allowed to penetrate the obstacle such that $u(t, l) - \varphi \le \varepsilon$ for all $t \in [0, T]$, where a sufficiently small $\varepsilon > 0$ is related to wear and hardness of its surface. If ε is too big, the obstacles will disintegrate from a physical point of view. We assume that there are two given quantities, $F = F(t, x)$, called a body force and $Q = Q(t, x)$, called an internal heat source.

As we mentioned above, we can expect four boundary conditions: two homogeneous Dirichlet boundary conditions on the left of the rod, $x = 0$, and two nonhomogeneous Neumann boundary conditions on the right of the rod, $x = l$. Our physical situation makes Neumann boundary conditions at $x = l$ to be two contact conditions; one is the normal compliance and another is Barber's heat exchange.

The normal compliance condition is defined with the positive part function, denoted by $(r)_+ = \max(r, 0)$ for $r \in \mathbb{R}$. Since the reactive obstacle is penetrated as mentioned above, contact forces $\sigma = \sigma(t)$ can be characterized by the power