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Dynamic contact of a beam against rigid obstacles: Convergence of a velocity-based approximation and numerical results

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This work is dedicated to the memory of J.J. Moreau

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A B S T R A C T

Motivated by the study of vibrations due to looseness of joints, we consider the motion of a beam between rigid obstacles. Due to the non-penetrability condition, the dynamics is described by a hyperbolic fourth order variational inequality. We build a family of fully discretized approximations of this problem by combining some classical space discretizations with velocity based time-stepping algorithms for discrete mechanical systems subjected to unilateral constraints. We prove the stability and the convergence of these numerical methods. Finally we propose some examples of implementation using either Hermite or B-spline finite element approximations.

1. Introduction

In many industrial devices, looseness of joints leads to unwanted vibrations which may cause untimely wear, defective behaviour or noise. Such bad consequences are often related to the occurrence of impacts between parts of the device. Motivated by this kind of problems, we will consider in this paper the dynamics of an elastic beam between rigid obstacles.

More precisely, we will denote by L the length of the beam and by $u(x,t)$, $(x,t) \in [0,L] \times [0,T]$, $T > 0$, the vertical displacement of a point x belonging to the beam axis. Then the shear stress is given by

$$\sigma = -k^2 u_{xxx}, \quad k^2 = \frac{EI}{\rho S}$$

where E and ρ are the Young's modulus and the density of the material and S and I are the surface and the inertial momentum of the cross section of the beam. See Fig. 1.

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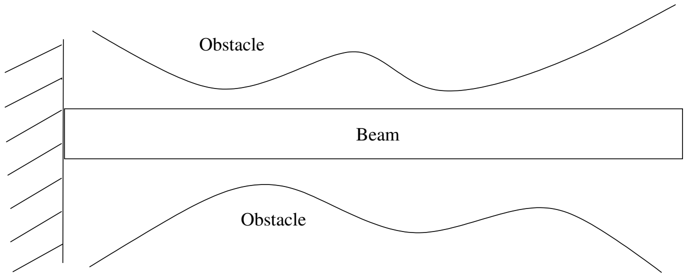

Fig. 1. Beam between longitudinal rigid obstacles.

The motion is limited by some rigid obstacles which position under and above the beam is characterized by two mappings $g_1, g_2 : [0, L] \to \mathbb{R} \cup \{-\infty, +\infty\}$ such that

$$g_1(x) \le -g < 0 < g \le g_2(x) \quad \forall x \in [0, L] \tag{1.1}$$

with $g \in \mathbb{R}^+_*$. Let us observe that g_1 and/or g_2 are not assumed to be continuous and $g_1(x)$ and/or $g_2(x)$ may take infinite values allowing to consider either longitudinal or punctual obstacles. The beam is clamped at its left end while it can move freely between the obstacles at its right end. Furthermore we assume that the motion is planar.

For the variational formulation of the problem, we introduce the following functional spaces:

$$H = L^2(0, L), \qquad V = \left\{ w \in H^2(0, L);\ w(0) = w_x(0) = 0 \right\}$$

and the convex set

$$K = \left\{ w \in V;\ g_1(x) \le w(x) \le g_2(x)\ \forall x \in [0, L] \right\}.$$

Then, for any given density of external forces $f \in L^2(0, T; H)$ and initial data $u_0 \in K$, $v_0 \in H$, the dynamics of the beam is described by (see [1] or [2,3])

$$(P) \begin{cases} \text{Find } u \in L^2(0, T; K) \text{ such that } u_t \in L^2(0, T; H),\ u(\cdot, 0) = u_0 \text{ and} \\ \quad -\displaystyle\int_0^T \int_0^L u_t(x, t)\big(w_t(x, t) - u_t(x, t)\big)\, dx\, dt + k^2 \int_0^T \int_0^L u_{xx}(x, t)\big(w_{xx}(x, t) - u_{xx}(x, t)\big)\, dx\, dt \\ \ge \displaystyle\int_0^L v_0(x)\big(w(x, 0) - u_0(x)\big)\, dx + \int_0^T \int_0^L f(x, t)\big(w(x, t) - u(x, t)\big)\, dx\, dt \\ \forall w \in L^2(0, T; K) \ \text{ such that } w_t \in L^2(0, T; H) \text{ and } w(\cdot, T) = u(\cdot, T). \end{cases}$$

For this kind of problem the first existence result has been obtained by K. Kuttler and M. Shillor [1] in the case of a punctual obstacle at $x = L$ i.e. $g_1(x) = -\infty$ and $g_2(x) = +\infty$ for all $x \in [0, L)$ and $-\infty < g_1(L) < 0 < g_2(L) < +\infty$ (see also [4] for the more complex geometrical setting of two beams in contact across a joint with clearance). For a longitudinal obstacle with a smooth shape located under the beam (i.e. $g_2(x) = +\infty$ for all $x \in [0, L]$ and $g_1 \in C^0\big([0, L]; \mathbb{R}^-\big)$ with $g_1(0) < 0$) another existence result is proposed by J. Ahn and D. Stewart in [5,6]. These results rely on the construction of approximate solutions either by using a normal compliance technique [1,6] or a time-discretization of the variational inequality [5]. The general case (i.e. any mappings g_1, g_2) is considered in [2] where fully discretized approximate problems are built, based on the method of lines, by combining a classical space discretization with a time-stepping algorithm for rigid bodies dynamics with unilateral constraints. Indeed, for any finite dimensional subspace V_h of V, the Galerkin approximation of (P) leads to

$$(P_h) \begin{cases} \text{Find } u_h \in L^2(0, T; K_h) \text{ such that } u_{h,t} \in L^2(0, T; H),\ u_h(\cdot, 0) = u_{h0} \text{ and} \\ \quad -\displaystyle\int_0^T \int_0^L u_{h,t}(x, t)\big(w_{h,t}(x, t) - u_{h,t}(x, t)\big)\, dx\, dt + k^2 \int_0^T \int_0^L u_{h,xx}(x, t)\big(w_{h,xx}(x, t) - u_{h,xx}(x, t)\big)\, dx\, dt \\ \ge \displaystyle\int_0^L v_{h0}(x)\big(w_h(x, 0) - u_{h0}(x)\big)\, dx + \int_0^T \int_0^L f(x, t)\big(w_h(x, t) - u_h(x, t)\big)\, dx\, dt \\ \forall w_h \in L^2(0, T; K_h) \ \text{ such that } w_{h,t} \in L^2(0, T; H) \text{ and } w_h(\cdot, T) = u_h(\cdot, T) \end{cases}$$

where K_h is a subset of V_h allowing to take into account the non-penetrability condition of the beam into the obstacles (see [2,3] or Section 2 for the detailed definition of K_h) and u_{h0} and v_{h0} are an approximation of u_0 and v_0 in V_h respectively. If

$$V_h = \mathrm{Span}\left\{\varphi_1, \varphi_2, \ldots, \varphi_{J_h}\right\} \subset V, \quad J_h = \mathrm{Dim}(V_h)$$

we can define

$$\mathcal{K}_h = \left\{ \bar{w}_h \in \mathbb{R}^{J_h} : w_h = \sum_{j=1}^{J_h} \bar{w}_h^j \varphi_j \in K_h \right\}.$$