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Existence for dynamic contact of a stochastic viscoelastic Gao Beam

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ABSTRACT

This work presents and analyzes a model for the vibrations of a viscoelastic Gao Beam, which may come in contact with a deformable random foundation and allows for stochastic inputs. The body force involves a stochastic integral that includes Brownian motion. In addition, the gap between the beam and the foundation is a stochastic process, which is one of the novelties in the paper, and contact is described with the normal compliance condition. The existence and uniqueness of strong solutions to the model is established and it is shown that the solutions are adapted to the filtration determined by a given Wiener process for the stochastic force noise term.

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1. Introduction

The Gao Beam is a one-dimensional nonlinear model of a moderately thin mechanical beam with a double-well elastic energy function that leads, under certain conditions, to two stable buckled steady states in addition to the unstable zero steady state. The model was derived in [1,2], see also [3], and various results on its simulations and analysis can be found in [4–8]. Unlike the usual models of linear beams, the Gao Beam can vibrate about the buckled states [4–6], which makes it a much more realistic and useful model for a beam or a supporting column. Problems of contact of the Gao Beam with a foundation were studied in [9,4,1].

In this work, we extend the dynamic deterministic Gao Beam model to include stochastic inputs of two kinds. First, the applied force is assumed to have also a noise component that is represented by a stochastic integral that includes Brownian motion. Secondly, the beam may come in contact with a reactive foundation and the gap between the foundation and the beam is assumed to be a stochastic process. These two novel extensions allow the Gao Beam model to better describe real vibrations when there is randomness in the system inputs, which invariably is the case in engineering applications. Although adding stochastic force terms to partial differential equations, thus dealing with stochastic equations, has considerable literature, the novelty here is that we add such a term to a fully nonlinear equation. The second novelty lies in allowing the shape of the obstacle to be random, described by a stochastic process, which is the first in the literature on contact mechanics.

The existence of weak or variational solutions to the Gao Beam, without randomness, when the beam may come in contact with an obstacle, usually called *the foundation*, was established in [9]. The solution was shown to be unique when the foundation was assumed to be reactive, modeled with the normal compliance condition, and the beam viscoelastic. The

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existence of a solution (but no uniqueness) were shown when the beam is elastic (in the inviscid limit) and in the case when the foundation is perfectly rigid. Here, we extend these results as mentioned above. First, we allow the applied force to have a stochastic component by adding a stochastic Ito integral. It is known that adding a stochastic input into a problem described by partial differential equations increases the complexity and the mathematical sophistication needed to analyze it. Moreover, it reduces considerably the regularity of the solutions, which precludes the use of many standard mathematical tools, see e.g., [10–14] and the references therein.

The addition of a stochastic process to describe the obstacle or the foundation is new. The peculiarity is that it makes the contact condition, which is nonlinear, also stochastic. The main result in this work, Theorem 9, asserts that up to an exceptional set of zero measure in the sample space Ω, the dynamic Gao Beam with stochastic inputs has a unique solution which is also adapted. The proof employs theorems on stochastic partial differential equations to obtain progressively measurable solutions to approximate problems which involve a truncation of the cubic term in the equation of motion. After this, we obtain estimates which allow us to obtain the solution for given ω by using a fixed point theorem. These estimates require the use of theorems from probability such as stopping times and the Burkholder–Davis–Gundy inequality. The usual techniques used in such problems cannot be applied directly. Then we use the uniqueness of the path solutions to show that there exists a progressively measurable solution path (outside of an exceptional set of zero measure).

The rest of the paper is structured as follows. Section 2 provides the necessary and substantial mathematical background and results used in the proofs. The description of the problems without and with stochastic inputs is provided in Section 3, and the variational or abstract formulation can be found in Section 4, where the statement of our main result, Theorem 9, is provided. The proof can be found in Section 5 and is based on the truncation of the cubic term, and establishing the existence and uniqueness of the solutions to the truncated problems, summarized in Theorem 10, and then obtaining the necessary estimates that allow us to pass to the limit without the truncation.

This work opens the way to study a variety of models of contact when stochastic effects and processes are taken into account. We foresee that this line of research will yield many important results in the near future.

2. Mathematical tools and results

In this section, which may be skipped on first reading, we present the necessary mathematical tools needed for the analysis of the model. We note that some of the results are new and have merit in and of themselves.

First, we describe shortly the *stochastic integral*. Then, we present two well-known embedding theorems that play crucial role in dealing with the contact processes. Finally, we provide some relevant theorems from probability theory that underlie the mathematical proofs.

2.1. The stochastic integral

The context or setting for the stochastic integral is briefly summarized here, further details can be found in [10,14] and the references therein. In all that follows the underlying probability space

$$(\Omega, \mathcal{F}_T, P)$$

is given. Here, Ω is the sample space, P is the probability function on Ω, and $\mathcal{F}_T$ is the last σ-algebra in a filtration $\mathcal{F}_t$, where $t \in [0, T]$. Thus, for each t, $\mathcal{F}_t$ is a σ-algebra and these σ-algebras are increasing in t. Also, we denote by ω an element of Ω.

It is assumed that the filtration is determined from a given Wiener process in the following way.

$$\mathcal{F}_t = \cap_{r>t} \overline{\sigma\ (W(s) : s \le r)}$$

where $\overline{\sigma\ (W(s) : s \le r)}$ denotes the completion of the smallest σ-algebra which has the property that $W(s)$ is measurable for each $s \le r$. It is given as an intersection so that the filtration has the property that $\mathcal{F}_t = \cap_{r>t} \mathcal{F}_r$. Thus, we first need to obtain a Wiener process. In infinite dimensions, this is a difficulty because it is defined as an infinite sum of independent weighted real Wiener processes and the sum tends to diverge. This is why the Wiener process and the stochastic integrals are given in terms of the diagram Fig. 1. In the diagram, we use the following conventions. U and H denote two separable Hilbert spaces and $L \in \mathcal{L}(U, H)$ is such that if $g, h \in LU$, we define $(g, h)_{LU} = \left(L^{-1}g, L^{-1}h\right)_U$, where $L^{-1}g$ is defined as $L\left(L^{-1}g\right) = g$ and $L^{-1}g$ is the closest to 0 in U out of all ϕ such that $L\phi = g$. It is somewhat analogous to the Moore–Penrose inverse of matrices in linear algebra.

We turn to diagram Fig. 1, in which U is a Hilbert space, possibly H itself, and J is a one-to-one Hilbert–Schmidt operator from the Hilbert space $Q^{1/2}U$ (just described) to the Hilbert space U_1 which could be U. Next, $Q \in \mathcal{L}(U, U)$ is a nonnegative and self-adjoint operator. We assume that $J : Q^{1/2}U \to U_1$ is a one-to-one Hilbert–Schmidt operator. It can be proved that such a Hilbert–Schmidt operator always exists. For the sake of completeness, we recall the definition of a Hilbert–Schmidt operator, using our context.

Definition 1. An operator $J : U \to H$ is said to be Hilbert–Schmidt if whenever $\{e_i\}_1^\infty$ is an orthonormal set in U,

$$\sum_{i=1}^{\infty} \|Je_i\|_H^2 < \infty.$$