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Two abstract mixed variational problems and applications in contact mechanics

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ABSTRACT

We consider two abstract mixed variational problems. Each of them consists of a system of two variational inequalities. The first problem involves two convex functionals while the second one involves a convex functional and a bifunctional which depends on a Lagrange multiplier in the first argument and is convex in the second argument. We obtain existence and uniqueness results for the first problem. Then, we combine the results we get with a fixed point technique in order to investigate the existence and the uniqueness of the solution of the second problem. The abstract results we obtain can be applied to the weak solvability of frictional contact models for nonlinearly elastic materials. To illustrate the applicability, three examples of frictional contact models are discussed.

1. Introduction

Having a strongly nonlinear feature, the contact models are always analyzed using techniques of nonlinear analysis. In particular, the weak solvability of contact models relies on techniques of variational calculus. One way is that one to deliver and analyze mixed variational formulations. Such kind of formulations are related to modern numerical techniques in order to approximate the weak solutions, see e.g. [1–4]. The efficient approximations motivate us to develop the mathematical framework of mixed variational problems. Recently, several papers were devoted to mixed variational problems with applications in contact mechanics, see e.g., [5–7]. The present paper brings a new contribution focusing on the following mixed variational problems on the Hilbert spaces X and Y.

First abstract problem. Given $f \in X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda \subset Y$ and

$$J(v) - J(u) + b(v - u, \lambda) + \phi(v) - \phi(u) \geq (f, v - u)_X \quad \text{for all } v \in X,$$
$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda.$$

Second abstract problem. Given $f \in X$, find $(u, \lambda) \in X \times Y$ such that $\lambda \in \Lambda \subset Y$ and

$$J(v) - J(u) + b(v - u, \lambda) + j(\lambda, v) - j(\lambda, u) \geq (f, v - u)_X \quad \text{for all } v \in X,$$
$$b(u, \mu - \lambda) \leq 0 \quad \text{for all } \mu \in \Lambda.$$

If Λ is a bounded subset of the space Y, the first abstract problem is a saddle point problem; see (3) for the corresponding functional. Otherwise, the first problem is a generalized saddle point problem. Also, the second abstract problem is a generalized saddle point problem. Notice that in the particular case $J : X \to \mathbb{R}\, J(v) = \frac{1}{2}a(v, v)$, $a : X \times X \to \mathbb{R}$ being

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a bilinear, symmetric, continuous, X-elliptic form, $\varphi \equiv 0$ and $j \equiv 0$, both problems reduce to the classical mixed variational problem

$$a(u, v) + b(v, \lambda) = (f, v)_X \quad \text{for all } v \in X,$$
$$b(u, \mu - \lambda) \le 0 \quad \text{for all } \mu \in \Lambda$$

which is a classical saddle point problem. In this particular case the corresponding functional is $\mathcal{L} : X \times \Lambda \to \mathbb{R}$ $\mathcal{L}(v, \mu) = \frac{1}{2}a(v, v) - (f, v)_X + b(v, \mu)$.

The main aim of this paper is to investigate the existence and the uniqueness of the solution for the proposed problems. Also, we are interested in their applicability in contact mechanics.

To start, we obtain existence and uniqueness results for the first problem. Then, combining these results with a fixed point technique, we investigate the existence and the uniqueness of the solution of the second problem.

Next, we illustrate the applicability of the abstract results to the weak solvability of three frictional contact models for nonlinearly elastic materials. The first one is related to the first abstract problem with unbounded set Λ, the second one is related to the first abstract problem with bounded set Λ while the third model leads to the second abstract problem.

From the mathematical point of view, the first problem has, indubitably, its own importance. But, at the same time, it can be viewed as an auxiliary problem in order to study the second abstract problem. However, from the mechanical point of view each problem has an area of applicability.

In applications we discuss in the present paper, the functional $J(\cdot)$ always depends on the constitutive map involved in the description of the behavior of the material, while the functional $\varphi(\cdot)$ and the bifunctional $j(\cdot, \cdot)$ depend on contact parameters.

The mathematical handling of the contact models requires complex knowledge of functional analysis and mathematical modeling in contact mechanics. Let us indicate a few references; see e.g. [8–11] for some useful mathematical tools and [12–22] for mathematical modeling in contact mechanics.

The rest of the paper has the following structure. Section 2 is dedicated to the analysis of the first abstract problem. Section 3 is devoted to the second abstract problem. In Section 4 we discuss the weak solvability of three contact models.

To increase the clarity of the exposure, we recall below the main tools we rely on.

Definition 1. Let A and B be two non-empty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to R$ if and only if

$$\mathcal{L}(u, \mu) \le \mathcal{L}(u, \lambda) \le \mathcal{L}(v, \lambda), \quad \text{for all } v \in A,\ \mu \in B.$$

Theorem 1. *Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be non-empty, closed, convex subsets. Assume that a real functional $\mathcal{L} : A \times B \to R$ satisfies the following conditions*

$$v \to \mathcal{L}(v,\ \mu) \text{ is convex and lower semicontinuous for all } \mu \in B,$$
$$\mu \to \mathcal{L}(v,\ \mu) \text{ is concave and upper semicontinuous for all } v \in A.$$

Moreover, assume that

$$A \text{ is bounded or } \lim_{\|v\|_X \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty \text{ for some } \mu_0 \in B$$

and

$$B \text{ is bounded or } \lim_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v,\ \mu) = -\infty.$$

Then, the functional $\mathcal{L}(\cdot, \cdot)$ has at least one saddle point.

For more details on the saddle point theory and its applications, we refer to [23–26].

In order to analyze the second abstract problem we shall use the following fixed point result.

Theorem 2. *Let E be a metrizable locally convex topological vector space and let E_C be a weakly compact convex subset of E. Then, any weakly sequentially continuous map $f : E_C \to E_C$ has a fix point.*

For the proof of Theorem 2, we refer to [27]. We recall that $f : E_C \to E_C$ is called a weakly sequentially continuous map if, for all sequences $(x_n)_n \subset E_C$, such that $x_n \rightharpoonup x$ in E then $f(x_n) \rightharpoonup f(x)$ in E.

We also recall the following embedding results.

Lemma 1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with Lipschitz continuous boundary $\partial\Omega$. Then, for $1 \le q < 4$, the operator $\gamma : H^1(\Omega) \to L^q(\partial\Omega)$ is completely continuous.*

For a proof of this result, we refer the reader to [28].

2. First abstract problem

The first section is devoted to the analysis of the following mixed variational problem.