Nonlinear Analysis: Real World Applications 22 (2015) 604–618

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

History-dependent variational–hemivariational inequalities in contact mechanics☆

Stanisław Migórski [a,*], Anna Ochal [a], Mircea Sofonea [b]

[a] Faculty of Mathematics and Computer Science, Jagiellonian University, ul. Łojasiewicza 6, 30348 Krakow, Poland

[b] Laboratoire de Mathématiques et Physique, Université de Perpignan Via Domitia, 52 Avenue Paul Alduy, 66860 Perpignan, France

ARTICLE INFO

Article history:
Received 9 April 2014
Received in revised form 2 September 2014
Accepted 25 September 2014
Available online 23 October 2014

Keywords:
Variational–hemivariational inequality
Clarke subdifferential
History-dependent operator
Viscoelastic material
Frictionless contact

ABSTRACT

We consider an abstract class of variational–hemivariational inequalities which arise in the study of a large number of mathematical models of contact. The novelty consists in the structure of the inequalities which involve two history-dependent operators and two non-differentiable functionals, a convex and a nonconvex one. For these inequalities we provide an existence and uniqueness result of the solution. The proof is based on arguments of surjectivity for pseudomonotone operators and fixed point. Then, we consider a viscoelastic problem in which the contact is frictionless and is modeled with a new boundary condition which describes both the instantaneous and the memory effects of the foundation. We prove that this problem leads to a history-dependent variational–hemivariational inequality in which the unknown is the displacement field. We apply our abstract result in order to prove the unique weak solvability of this viscoelastic contact problem.

1. Introduction

Variational and hemivariational inequalities play an important role in the study of both the qualitative and numerical analysis of nonlinear boundary value problems arising in Mechanics, Physics and Engineering Science. At the heart of this theory is the intrinsic inclusion of free boundaries in an elegant mathematical formulation.

The study of variational inequalities started in early sixties, based on arguments of monotonicity and convexity, including the properties of the subdifferential of a convex function. Basic references concerning the theory of variational inequalities are [1–9]. The numerical analysis of various classes of variational inequalities was treated in [10,11] and also in [12,13]. For various applications of the variational inequalities in Mechanics and Engineering Science we refer to [9,14,15], among others. An excellent reference in the study of numerical analysis of plasticity problems via the theory of variational inequalities is [16].

The notion of hemivariational inequality was introduced in [17], based on the properties of generalized gradient introduced and studied in [18–20]. The Clarke theory of subdifferentiation for locally Lipschitz functions has been motivated by

☆ Research was supported by the Marie Curie International Research Staff Exchange Scheme Fellowship within the 7th European Community Framework Programme under Grant Agreement No. 295118 and the National Science Center of Poland under the Maestro Advanced Project No. DEC-2012/06/A/ST1/00262. The first two authors are also partially supported by the National Science Center of Poland under Grant No. N N201 604640 and the International Project co-financed by the Ministry of Science and Higher Education of Republic of Poland under Grant No. W111/7.PR/2012, and the project Polonium "Mathematical and Numerical Analysis for Contact Problems with Friction" 2014/15 between the Jagiellonian University in Krakow and Université de Perpignan Via Domitia.

* Corresponding author. Tel.: +48 12 664 6666; fax: +48 12 664 6673.
E-mail address: Migorski@ii.uj.edu.pl (S. Migórski).

http://dx.doi.org/10.1016/j.nonrwa.2014.09.021
1468-1218/

the facts that a convex function is locally Lipschitz in the interior of its effective domain and a locally Lipschitz function is differentiable almost everywhere. The analysis of hemivariational inequalities, including existence and uniqueness results, can be found in [21–24]. For a description of various problems arising in Mechanics and Engineering Science which lead to hemivariational inequalities we refer to [22,24]. For the numerical analysis of hemivariational inequalities, including their finite element approximation, we refer to [25].

Variational–hemivariational inequalities represent a special class of inequalities, in which both convex and nonconvex functions are involved. Their study is motivated by various boundary problems arising in Mechanics and, more specifically, in Contact Mechanics. Basic references in the field are [23,24].

Contact phenomena involving deformable bodies abound in industry and everyday life and, owing to their inherent complexity, they lead to nonlinear and nonsmooth mathematical problems. Their mathematical and numerical analysis, including existence and uniqueness results, was developed in a large number of works, see for instance the books [9,14,15,22,26], the edited volumes [27–31] and the references therein. It is based on arguments of variational and hemivariational inequalities, among others.

The aim of this paper is to prove a new existence and uniqueness result in the study of a class of variational–hemivariational inequalities and to apply these results in the analysis of a quasistatic contact problem. Our work has three traits of novelty which makes the difference with our previous paper [32] where the unique solvability of a class of history-dependent hemivariational inequalities was proved. The first novelty arises in the special structure of the abstract problems we consider, which are governed by two operators depending on the history of the solution, and two nondifferentiable functionals, a convex and a nonconvex one. Therefore, in contrast to [32], in this paper we deal with variational–hemivariational inequalities. The second novelty arises in the fact that, unlike a large number of references, including [32], the inequality problems considered in this paper are formulated on the unbounded interval of time $\mathbb{R}_+ = [0, \infty)$. Finally, the third one is given by the contact model we consider which describes a reactive foundation with memory effects. Considering such kind of models leads to a new and nonstandard mathematical model.

The class of variational–hemivariational inequalities we study represents a general framework in which a large number of quasistatic contact problems, associated to various constitutive laws and frictional or frictionless contact conditions, can be cast. Therefore, our work provides arguments and tools which can be useful to prove the unique solvability of a large number of quasistatic contact problems.

The rest of the paper is structured as follows. In Section 2 we recall some preliminary material. In Section 3 we introduce the class of history-dependent variational–hemivariational inequalities we are interested in, list the assumptions on the data and state our main existence and uniqueness result, Theorem 16. The proof of the theorem is presented in Section 4. It is based on arguments of surjectivity for pseudomonotone operators and fixed point. In Section 5 we introduce a quasistatic frictionless contact problem in which the material's behavior is modeled with a viscoelastic constitutive law and the contact conditions are in a subdifferential form. We show that this problem leads to a history-dependent variational–hemivariational inequality for the displacement field. Then, in Section 6, we use our abstract result in the analysis of this contact problem and prove its unique solvability.

2. Preliminaries

We start with presenting some preliminary material which will be used in the rest of the paper. It includes the definitions of classes of single and multivalued operators and the properties of subgradients of convex functions and locally Lipschitz functions. For further details, we refer the reader to the books [21,22,33].

Given a normed Banach space X we denote by $\|\cdot\|_X$ its norm, by X^* its topological dual and by $\langle\cdot,\cdot\rangle_{X^*\times X}$ the duality pairing of X and X^*. The symbol w-X is used for the space X endowed with the weak topology. Moreover, 2^{X^*} represents the set of parts of X^*. For simplicity in exposition, in the following, we always suppose that X is a Banach space unless otherwise stated. Below we shall consider both single-valued operators $A: X \to X^*$ and multivalued operators $A: X \to 2^{X^*}$. The following definitions hold for single-valued operators.

Definition 1. An operator $A: X \to X^*$ is called:

(a) *bounded*, if A maps bounded sets of X into bounded sets of X^*.
(b) *monotone*, if $\langle Au - Av, u - v\rangle_{X^*\times X} \ge 0$ for all $u, v \in X$.
(c) *maximal monotone*, if it is monotone, and $\langle Au - w, u - v\rangle_{X^*\times X} \ge 0$ for any $u \in X$ implies that $w = Av$.
(d) *coercive with constant* $\alpha > 0$, if $\langle Au, u\rangle_{X^*\times X} \ge \alpha \|u\|_X^2$ for all $u \in X$.
(e) *pseudomonotone*, if it is bounded and $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \le 0$ imply $\langle Au, u - v\rangle_{X^*\times X} \le \liminf \langle Au_n, u_n - v\rangle_{X^*\times X}$ for all $v \in X$.

Remark 2. An equivalent definition of pseudomonotonicity which is useful in the following, reads as follows. An operator $A: X \to X^*$ is pseudomonotone, if it is bounded, and if $u_n \to u$ weakly in X together with $\limsup \langle Au_n, u_n - u\rangle_{X^*\times X} \le 0$ imply $\lim \langle Au_n, u_n - u\rangle_{X^*\times X} = 0$ and $Au_n \to Au$ weakly in X^*.