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# A bending–stretching model in adhesive contact for elastic rods obtained by using asymptotic methods

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ABSTRACT

This paper is devoted to the derivation and mathematical justification of models for the bending–stretching of an elastic rod in adhesive contact with a deformable foundation. The process is assumed to be quasistatic, and therefore the effects of inertia are neglected. Contact is modeled with normal compliance and the adhesion is modeled by introducing a surface internal variable, the bonding function, the evolution of which is described by an ordinary differential equation. To derive the models we consider the three-dimensional contact problem of an elastic body in adhesive contact with a foundation, introduce a change of variable together with the scaling of the unknowns and parameters of the problem, and we obtain a limit model under the assumption of suitable asymptotic expansions for the scaled unknowns. After that, we obtain error estimates and convergence results which legitimate the limit model. Finally we show that our limit model contains as particular cases models previously considered by other authors. To our knowledge it is for the first time that a rigorous justification and a generalization of those models is provided.



## 1. Introduction

In solid mechanics, the obtention of models for rods, beams, bars, plates and shells is based on *a priori* hypotheses on the displacement and/or stress fields which, upon substitution in the equilibrium and constitutive equations of three-dimensional elasticity, lead to useful simplifications. Nevertheless, from both constitutive and geometrical points of views, there is a need to justify the validity of most of the models obtained in this way.

For this reason a considerable effort has been made in the past decades by many authors in order to derive new models and justify the existing ones by using the asymptotic expansion method, whose foundations can be found in [1]. Indeed, the first applied results were obtained with the justification of the linearized theory of plate bending in [2,3].

The theories of beam bending and rod stretching also benefited from the extensive use of asymptotic methods and so the justification of the Bernoulli–Navier model for the bending–stretching of elastic thin rods was provided in [4]. In the following years, the nonlinear case was studied in [5] and the analysis and error estimation of higher-order terms in the asymptotic expansion of the scaled unknowns was given in [6]. In [7], the authors use the asymptotic method to justify the Saint-Venant, Timoshenko and Vlassov models of elastic beams. More recently, several models for the bending–stretching of viscoelastic rods were provided in [8,9] and the derivation of models for elastic rods on contact with a deformable foundation has been developed, first for a frictionless case in [10] and after that, for a frictional model including wear in [11].

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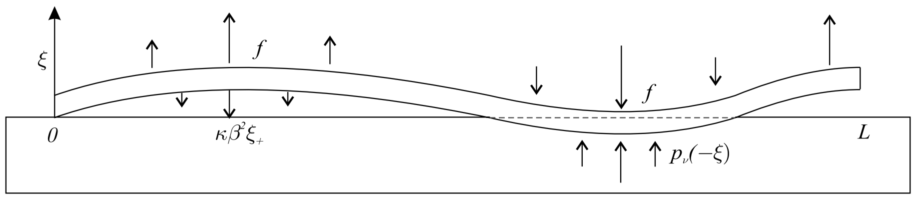


**Fig. 1.** Physical setting.

Contact phenomena involving deformable bodies abound in industry and everyday life. The contact of braking pads with wheels, a tire with a road, a piston with a skirt, a shoe with a floor, are just a few simple examples. For this reason, considerable progress has been made with the modeling and analysis of contact problems, and the engineering literature concerning this topic is rather extensive. An early attempt to the study of frictional contact problems within the framework of variational inequalities was made in [12]. Comprehensive references on analysis and numerical approximation of variational inequalities arising from contact problems include [13–15]. Mathematical, mechanical and numerical state of the art on the Contact Mechanics can be found in the proceedings [16,17], in the special issue [18] and in the monograph [19], as well.

The adhesive contact between deformable bodies, when a glue is added to prevent relative motion of the surfaces, has received increased attention in the mathematical literature. Basic modeling can be found in [20–23]. Analysis of models for adhesive contact can be found in [24–28].

In all the above papers the adhesion is modeled by the introduction of a surface internal variable, the bonding field, denoted in this paper by $\beta$; it describes the pointwise fractional density of active bonds on the contact surface, and sometimes referred to as the intensity of adhesion. Following [20,21], the bonding field satisfies the restrictions $0 \le \beta \le 1$; when $\beta = 1$ at a point of the contact surface, the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction $\beta$ of the bonds is active. We refer the reader to the extensive bibliography on the subject in [22,19,29].

Models of beams in frictionless contact with adhesion, both in the dynamic and the quasistatic case, can be found in [30]. Nevertheless, that paper focuses on the analysis of the models, including existence and uniqueness results for the weak solutions, without providing explanation on how the corresponding models were derived. Therefore, despite the progress made in that paper, there is a real need to justify such kind of models of adhesive contact involving thin structures.

The aim of the present paper is to contribute to the filling of this gap. More precisely, we derive models for the adhesive contact of an elastic rod, by using the asymptotic expansion method and, to the best of our knowledge, this is for the first time that such kind of models are rigorously derived. We obtain as a particular case of our main result, Theorem 5.1, the following model for the vertical separation, $\xi$, of an elastic rod from an obstacle when they are in adhesive contact:

$$\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 \xi}{\partial x^2}\right) = f + |\gamma_C|\left(p(-\xi) - \kappa\beta^2\xi_+\right) \quad \text{in } (0,L)\times(0,T), \tag{1.1}$$

$$\dot{\beta} = -\left(\delta\beta(\xi_+)^2 - \epsilon_a\right)_+ \quad \text{in } (0,L)\times(0,T). \tag{1.2}$$

Here $\beta$ denotes the bonding function (the intensity of adhesion), the model allows penetration, the reaction given by the normal compliance function $p$ (which satisfies $p(r) = 0$ if $r \le 0$), $\kappa$ is the adhesion coefficient, $\delta$ is the adhesion rate, and $f$ represents external loads. Also, $\gamma_C$ is the contact boundary of the cross section, $\epsilon_a$ is a threshold for the bonds to start degradation, $E$ is Young's modulus of the material, $I$ is the inertia moment for the cross section, $L$ is the length of the rod and $T$ is the period of observation (see Fig. 1).

We remark that the system (1.1)–(1.2) is a model found in the literature (see for example [30]), and our goal consists in providing a mathematical justification for it. A more general situation is obtained by considering the obstacle described by a function $y = \phi(x)$. We only consider here the particular case $\phi = 0$ for the sake of simplicity.

The rest of the paper is organized as follows. In Section 2 we introduce some notations and preliminary material and recall the existence and uniqueness of solution of the three-dimensional elasticity contact problem (which was first given in [28]), formulated in the volume $\Omega^\varepsilon$ occupied by the rod, $\varepsilon$ being the size of the diameter of the transversal section $\omega^\varepsilon$. The unknowns are denoted $\boldsymbol{u}^\varepsilon$ and $\beta^\varepsilon$, which represent the displacement and the bonding function, respectively. The main ingredient in our approach is developed in Section 3, and it consists in introducing a change of variable together with the scaling of the unknowns. In this way, the problem is reduced to an equivalent one, formulated in a reference domain $\Omega$, with contact boundary $\Gamma_C$. The unknowns of this new problem are denoted as $\boldsymbol{u}(\varepsilon)$ and $\beta(\varepsilon)$, and represent the scaled displacement and the scaled bonding, respectively, for which we assume asymptotic expansions. The mathematical justification of the model is provided in Section 4, and it is supported by error estimates and a convergence result of the form $\boldsymbol{u}(\varepsilon) \to \boldsymbol{u}^0$ in $[H^1(\Omega)]^3$ and $\beta(\varepsilon) \to \beta^0$ in $L^2(\Gamma_C)$, where $\boldsymbol{u}^0 = (u_i^0)$ and $\beta^0$ are the first order terms of the respective asymptotic expansions. Finally, in Section 5, after the "descaling" of $\{\boldsymbol{u}(\varepsilon), \beta(\varepsilon)\}$, which gives an asymptotic expansion of $\{\boldsymbol{u}^\varepsilon, \beta^\varepsilon\}$, we characterize the zeroth order term of such expansions in terms of the solution of a problem, which describes the axial deformation and bendings of an elastic beam in frictionless adhesive contact with a deformable foundation.