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The interaction of rarefaction waves of a two-dimensional nonlinear wave system

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ABSTRACT

This paper is concerned with a two-dimensional Riemann problem for the nonlinear wave system which gives rise to an interaction of two planar rarefaction waves. This problem comes from the expansion of a wedge of gas with constant velocity into vacuum, which is often interpreted as the dam collapse problem in hydraulics. We establish the global existence of smooth solutions in the interaction region of rarefaction waves up to a non-trivial vacuum boundary.

1. Introduction

In this paper, we are interested in a special two-dimensional Riemann problem to the nonlinear wave system:

$$\begin{cases} \rho_t + (\rho u)_x + (\rho v)_y = 0, \\ (\rho u)_t + p_x = 0, \\ (\rho v)_t + p_y = 0, \end{cases} \tag{1.1}$$

where $\rho \geq 0$ is the density, (u, v) is the velocity and p is the pressure given by $p(\rho) = A\rho^\gamma$ where $A > 0$ will be scaled to be one and $\gamma > 1$ is the gas constant. System (1.1) is obtained either by ignoring the quadratic terms in the velocity (u, v) to the two-dimensional isentropic compressible Euler equations in gas dynamics, or by writing the nonlinear wave equation as a first-order system, see [1,2] for more background information.

The study of two-dimensional Riemann problem to the Euler equations was initiated by Zhang and Zheng [3]. Based on the generalized characteristic analysis method and numerical experiments, a set of conjectures on the structure of solutions were presented. Unfortunately, until now, none of them has been completely proved due to the complicated structure of solutions. Many efforts have been made to understand these constructions for specific equations such as the potential flow equation, the unsteady transonic small disturbance equations, the pressure-gradient system and the nonlinear wave equation. We refer the reader to the monographs [4,5] and the survey [6] and references therein for the relevant results about the two-dimensional Riemann problem to these models. In particular, for the nonlinear wave system (1.1), Jegdic, Keyfitz and Čanié [7] established the existence of local solutions to the regular transonic shock reflection problem; Tesdall, Sanders and Keyfitz [8] provided numerical results for the von Neumann triple point paradox; Kim [9] and Kim and Lee [10] studied the configuration that the transonic shock interacted with the sonic circle and obtained a global transonic solution

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to this configuration. Recently, Kim [11] has constructed a global transonic solution to the interaction of a transonic shock with a rarefaction wave and Chen et al. [12] have established a global theory of existence and optimal regularity for a shock diffraction problem for system (1.1).

In this paper, we consider the expansion problem of a wedge of gas into vacuum for system (1.1), which is often interpreted as the dam collapse problem in hydraulics [13,14]. In the context of two-dimensional Riemann problems, the gas expansion problem is that of the interaction of two two-dimensional planar rarefaction waves [15]. The first global result about the interaction of rarefaction waves was given by Li [16] for the isothermal Euler equations in the hodograph plane. The cases of the isentropic Euler equations were treated first in [17] and then completed subsequently in [15,18–20]. Similar problems for the pressure-gradient system were discussed by Dai and Zhang [21] and Lei and Zheng [22]. For more related papers, see [23–31].

The purpose of the present paper is to construct global classical solutions to the interaction of two arbitrary planar rarefaction waves for system (1.1). Since the interaction domain is bounded by two characteristics and has the vacuum point as the degenerate point, then the problem is a degenerate Goursat-type boundary value problem. In order to solve this degenerate Goursat problem, we derive an interesting characteristic decomposition for the pressure, see (3.1), to establish a priori C^0, C^1 and $C^{1,1}$-norms estimates. Due to the degeneracy, the C^1 and $C^{1,1}$-norms estimates are not uniform. We employ the ideal used by Dai and Zhang [21] for the pressure-gradient system to extend the local solution to the global one.

1.1. Setup of the problem

For simplicity, we place the wedge symmetrically with respect to the x-axis and the sharp corner at the origin, as in Fig. 1(a). Then the problem is formulated mathematically as seeking a solution of (1.1) with the initial data

$$(p, m, n)(0, x, y) = \begin{cases} (p_0, m_0, n_0), & -\theta_0 < \theta < \theta_0, \\ (0, \bar{m}, \bar{n}), & \text{otherwise}, \end{cases} \tag{1.2}$$

where $(m, n) = (\rho(p)u, \rho(p)v)$ is the momentum, $p_0 > 0$, m_0 and n_0 are constants, $(\bar{m}, \bar{n})$ is the momentum of the wave front, not being specified in the state of vacuum, $\theta = \arctan(y/x)$ is the polar angle, and θ_0 is the half-angle of the wedge restricted between 0 and $\pi/2$. For convenience, we assume that initially the gas is at rest, i.e., $(u_0, v_0) = (0, 0)$ or $(m_0, n_0) = (0, 0)$. After the edges L_1 and L_2 are removed, the gas away from the sharp corner expands into the vacuum as planar rarefaction waves R_1 and R_2, which are, respectively, explicitly expressed in the self-similar plane $(\xi, \eta) = (x/t, y/t)$ as

$$R_1: \begin{cases} p = \gamma^{\frac{-1}{\kappa}} (\sin\theta_0\xi - \cos\theta_0\eta)^{\frac{2}{\kappa}}, \\ m = \dfrac{2\sqrt{\gamma}}{\gamma+1}\sin\theta_0\left\{\gamma^{\frac{\kappa-2}{2\kappa}}(\sin\theta_0\xi - \cos\theta_0\eta)^{\frac{2-\kappa}{\kappa}} - p_0^{2-\kappa}\right\}, \\ n = -\dfrac{2\sqrt{\gamma}}{\gamma+1}\cos\theta_0\left\{\gamma^{\frac{\kappa-2}{2\kappa}}(\sin\theta_0\xi - \cos\theta_0\eta)^{\frac{2-\kappa}{\kappa}} - p_0^{2-\kappa}\right\}, \\ 0 \le \sin\theta_0\xi - \cos\theta_0\eta \le \sqrt{\gamma}p_0^{\frac{\kappa}{2}}, \end{cases}$$

and

$$R_2: \begin{cases} p = \gamma^{\frac{-1}{\kappa}} (\sin\theta_0\xi + \cos\theta_0\eta)^{\frac{2}{\kappa}}, \\ m = \dfrac{2\sqrt{\gamma}}{\gamma+1}\sin\theta_0\left\{\gamma^{\frac{\kappa-2}{2\kappa}}(\sin\theta_0\xi + \cos\theta_0\eta)^{\frac{2-\kappa}{\kappa}} - p_0^{2-\kappa}\right\}, \\ n = \dfrac{2\sqrt{\gamma}}{\gamma+1}\cos\theta_0\left\{\gamma^{\frac{\kappa-2}{2\kappa}}(\sin\theta_0\xi + \cos\theta_0\eta)^{\frac{2-\kappa}{\kappa}} - p_0^{2-\kappa}\right\}, \\ 0 \le \sin\theta_0\xi + \cos\theta_0\eta \le \sqrt{\gamma}p_0^{\frac{\kappa}{2}}, \end{cases}$$

where $\kappa = (\gamma-1)/\gamma \in (0, 1)$ by $\gamma > 1$. At the beginning, R_1 and R_2 interact at $M_0 = (\frac{\sqrt{\gamma p_0^{\kappa}}}{\sin\theta_0}, 0)$, and then they are separated from a region, the wave interaction region Ω, by two characteristics l_- and l_+ from the point M_0, see Fig. 1(b). The solution outside Ω consists of the constant state $(p_0, 0, 0)$, the planar rarefaction waves R_1, R_2 and the vacuum. The problem is to seek a solution of (1.1) inside the wave interaction region Ω subject to the boundary conditions on $l_\pm$. Since both $l_\pm$ are characteristics and $(\xi, \eta) = (0, 0)$ is a vacuum point, then the problem is a degenerate Goursat-type boundary value problem.

1.2. The main results

In terms of self-similar variables (ξ, η), the nonlinear wave system (1.1) becomes

$$\begin{cases} -\dfrac{1}{\gamma}\xi p^{-\kappa}p_\xi - \dfrac{1}{\gamma}\eta p^{-\kappa}p_\eta + m_\xi + n_\eta = 0, \\ -\xi m_\xi - \eta m_\eta + p_\xi = 0, \\ -\xi n_\xi - \eta n_\eta + p_\eta = 0, \end{cases} \tag{1.3}$$