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The Maxey–Riley equation: Existence, uniqueness and regularity of solutions

Mohammad Farazmand [a,b,*], George Haller [b]

[a] Department of Mathematics, ETH Zurich, Rämistrasse 101, 8092 Zurich, Switzerland
[b] Institute of Mechanical Systems, ETH Zurich, Tannenstrasse 3, 8092 Zurich, Switzerland

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ABSTRACT

The Maxey–Riley equation describes the motion of an inertial (i.e., finite-size) spherical particle in an ambient fluid flow. The equation is a second-order, implicit integro-differential equation with a singular kernel, and with a forcing term that blows up at the initial time. Despite the widespread use of the equation in applications, the basic properties of its solutions have remained unexplored. Here we fill this gap by proving local existence and uniqueness of mild solutions. For certain initial velocities between the particle and the fluid, the results extend to strong solutions. We also prove continuous differentiability of the mild and strong solutions with respect to their initial conditions. This justifies the search for coherent structures in inertial flows using the Cauchy–Green strain tensor.

1. Introduction

The Maxey–Riley equation [1–3] describes the motion of a small but finite-sized rigid sphere through a fluid. The equation is widely used to study the motion of a finite-size (or inertial) particle immersed in a non-uniform fluid. The behavior of such particles is of interest in various environmental and engineering problems, e.g., clustering of garbage patches in the oceans [4] and dispersion of airborne pollutants [5].

A first attempt to derive the equation of motion of an inertial particle in a non-uniform flow appears in [1]. Tchen [1] wrote the Basset–Boussinesq–Oseen equation (governing the motion of a small spherical particle in a quiescent fluid [6–8]) in a frame co-moving with a fluid parcel in an unsteady flow, accounting for various forces that arise in such non-inertial frames. Later, the exact form of the forces exerted on the particle were debated and corrected by several authors (see, e.g., [2,3,9]). Today, the most widely accepted form of the equations is the Maxey–Riley (MR) equation [3] with the corrections due to Auton et al. [9] and Maxey [10].

To recall the exact form of the MR equation, we let $u : \mathcal{D} \times \mathbb{R}^+ \to \mathbb{R}^n$ denote the velocity field describing the flow of a fluid in an open spatial domain $\mathcal{D} \subseteq \mathbb{R}^n$, where $n = 2$ or $n = 3$ for two- or three-dimensional flows, respectively. A fluid trajectory is then the solution of the differential equation $\dot{x} = u(x, t)$ with some initial condition $x(t_0) = x_0$. A spherical inertial particle, however, follows a different trajectory $y(t) \in \mathcal{D}$, which satisfies the MR equation

$$\ddot{y} = \frac{R}{2}\frac{\mathrm{D}}{\mathrm{D}t}\left(3u(y,t) + \frac{\gamma}{10}\mu^{-1}\Delta u(y,t)\right) + \left(1 - \frac{3R}{2}\right)g - \mu\left(\dot{y} - u(y,t) - \frac{\gamma}{6}\mu^{-1}\Delta u(y,t)\right)$$
$$- \kappa\mu^{1/2}\left\{\int_{t_0}^{t} \frac{\dot{w}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{w(t_0)}{\sqrt{t-t_0}}\right\}, \tag{1}$$

* Corresponding author at: Institute of Mechanical Systems, ETH Zurich, Tannenstrasse 3, 8092 Zurich, Switzerland. Tel.: +41 446329987.
E-mail address: farazmam@ethz.ch (M. Farazmand).

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where

$$w(t) = \dot{y}(t) - u(y(t), t) - \frac{\gamma}{6}\mu^{-1}\Delta u(y(t), t). \tag{2}$$

The initial conditions for the inertial particle are given as $y(t_0) = y_0$ and $\dot{y}(t_0) = v_0$. The material derivative $\frac{\mathrm{D}}{\mathrm{D}t} \doteq \partial_t + u \cdot \nabla$ denotes the time derivative along a fluid trajectory.

All the variables and parameters in Eqs. (1) and (2) are dimensionless, nondimensionalized by characteristic length scale L, characteristic velocity U and characteristic time scale $T = L/U$ of the fluid flow. The dimensionless parameters are

$$R = \frac{2\rho_f}{\rho_f + 2\rho_p}, \qquad \mu = \frac{R}{St}, \qquad \kappa = \sqrt{\frac{9R}{2\pi}}, \qquad \gamma = \frac{9R}{2\mathrm{Re}}, \tag{3}$$

where ρ_f and ρ_p are the density of the fluid and the particle, respectively. The constant, dimensionless vector of gravity is denoted by g. The Stokes (St) and Reynolds (Re) numbers are defined as

$$\mathrm{St} = \frac{2}{9}\left(\frac{a}{L}\right)^2 \mathrm{Re}, \quad \mathrm{Re} = \frac{UL}{\nu}, \tag{4}$$

where a is the radius of the particle and ν denotes the kinematic viscosity of the fluid.

Eq. (1) is a system of nonlinear, fractional-order differential equations. The fractional order is due to the *memory term*

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t} \frac{w(s)}{\sqrt{t-s}}\mathrm{d}s = \int_{t_0}^{t} \frac{\dot{w}(s)}{\sqrt{t-s}}\mathrm{d}s + \frac{w(t_0)}{\sqrt{t-t_0}} \tag{5}$$

where the identity is obtained by subsequent differentiation and integration-by-part (see, e.g., [11]). The memory term is a fractional derivative of order 1/2 in the Riemann–Liouville sense [11,12]. Physically, it represents the Basset–Boussinesq force [6,7,13] resulting from the lagging boundary layer development around the particle, as it moves through the fluid [3].

In the original derivation of the MR equation [3], it is implicitly assumed that the initial velocity of the particle v_0 is such that $w(t_0) = 0$ holds. Eq. (1), however, is the most general form of the MR equation which was derived later [10] and allows for a general initial particle velocity v_0.

Without the memory term and for $w(t_0) = 0$, the MR equation is an ordinary differential equation, whose solutions are well known to be regular for any smooth ambient velocity field $u(x, t)$. The memory term, however, introduces complications in the analysis and numerical solution of the equation. It contains an implicit term through the integral with an integrand depending on the particle acceleration $\ddot{y}$. Because of its implicit nature, it is not a priori clear if the MR differential equation defines a dynamical system, i.e., a process with a well-defined flow map.

Furthermore, when nonzero, the unbounded term $w(t_0)/\sqrt{t-t_0}$ further complicates Eq. (1), imparting an instantaneously infinite force at the initial time. This term is often ignored for convenience, even though its omission imposes a special constraint on the initial particle velocity that is hard to justify physically [10].

For the above reasons, the memory term has routinely been neglected in studies of inertial particle dynamics (see, e.g., Maxey [14], Babiano et al. [15], Haller and Sapsis [16]), until recent studies demonstrated convincingly the quantitative and qualitative importance of the memory term (see, e.g., [17–19] for experimental and [20–22] for numerical studies).

In addition to theoretical difficulties, the memory term also complicates the numerical treatment of the full MR equation. This equation is certainly not solvable with standard numerical schemes such as Runge–Kutta algorithms. To this end, involved schemes have been developed for numerical treatment of the memory term (see Daitche [12] and references therein).

All these numerical schemes implicitly assume the existence and uniqueness of solutions of the MR equation. The solutions can indeed be found explicitly for certain simple velocity fields [17,23]. To the best of our knowledge, however, general existence and uniqueness results have not been proven, and cannot be directly concluded from existing results on broader classes of evolution equations (see [11,24–26] for related but not applicable results on integro- and fractional-order differential equations). In the absence of such results, the existence and regularity of solutions for a nonlinear system of fractional-order differential equation, such as the MR equation, is far from obvious.

Here, we present the first proof of local existence and uniqueness of mild solutions to the full MR equation. The solutions become classical (strong) solutions to (1) for initial conditions satisfying $w(t_0) = 0$. Moreover, we show that both the mild and the strong solutions are continuously differentiable with respect to their initial conditions. As a consequence, coherent-structure detection methods utilizing the derivative of the flow map in the absence of the memory term [27] can also be employed in the present, more general context.

We start with re-writing the MR equation as a system of differential equations (see Eq. (8)) in terms of the particle position y and the function w defined in (2). Multi-dimensional reformulations of the MR equation have appeared before [12,23,28] but remained inaccessible to general mathematical analysis due to the implicit dependence of their right-hand sides on $\dot{y}$.

Our formulation turns the MR equation into a nonlinear system of fractional-order differential equations in terms of y and w. The standard techniques for the proof of existence and uniqueness of solutions of such equations assume Lipschitz continuity of the right hand side with respect to the (y, w) variable [11,26]. This assumption fails for the MR equation (see the term $M_u(y, t)w$ in Eq. (8)). Therefore, as discussed in Section 3, modifications to the standard function spaces, estimates and assumptions are required.