Nonlinear Analysis: Real World Applications 22 (2015) 107–114

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Existence of solitary waves in nonlocal nematic liquid crystals

Guoqing Zhang [a,*], Zhonghai Ding [b]

[a] College of Sciences, University of Shanghai for Science and Technology, Shanghai 200093, PR China

[b] Department of Mathematical Sciences, University of Nevada Las Vegas, Las Vegas, NV 89154-4020, USA

ARTICLE INFO

Article history:
Received 3 January 2014
Received in revised form 14 August 2014
Accepted 22 August 2014
Available online 15 September 2014

Keywords:
Solitary waves
Nonlocal nematic liquid crystals
Variational methods

ABSTRACT

In this paper, we study the solitary waves in nonlocal nematic liquid crystals. By applying the Mountain Pass Theorem and the Krasnoselskii genus theory, we prove some existence and multiplicity results of radially symmetric solitary waves.

1. Introduction

Liquid crystals are organic mesophases featuring various degrees of spatial order while retaining basic properties of fluid. Nematic liquid crystals (NLCs) are common materials found in many consumer electronic devices. NLCs display the long-range order of crystals, and contain rod-like molecules exhibiting both orientational alignment without position order and strong optical nonlinearities due to large refractive index anisotropy. Another important property of NLCs is the ability to change optical characteristics when an external electric field is applied and enables the macroscopic reorientation of the director tilt angle. Since the light incident on a NLC modifies the electric permittivity tensor and leads to reorientation nonlinearity, the study of propagation of self-focused beams in NLCs have attracted a lot of attentions from engineers and physicists in recent years. The optical spatial solitons in NLCs, termed as nematicons, have been observed experimentally [1–4] and numerically [5–9]. For the recent developments of nematicons, the reader is referred to an excellent review paper by Peccianti and Assanto [10].

The distortion of molecular orientation in NLCs can be described by the reorientation angle θ of the director with respect to the z-axis. In the presence of an external low frequency electric field, the spatial evolution of a slowly-varying beam envelope $E(x, y, z)$, which is linearly polarized along the x-axis and propagates along the z-axis, is governed by the nonlinear Schrödinger-like paraxial wave equation. The molecular orientation angle θ is governed by the elliptic equation. The model equations for the optical field (E, θ) are given by [1,3,8]

$$2i\frac{\partial E}{\partial z} + \Delta_{x,y}E + \alpha[\sin^2\theta - \sin^2\theta_0]E = 0, \tag{1.1}$$

$$2\Delta_{x,y}\theta + [\beta + \alpha|E|^2]\sin(2\theta) = 0, \tag{1.2}$$

where θ_0 is the pretilt angle which is the orientation induced only by the static electric field, $\Delta_{x,y}$ is the Laplacian, α and β denote the optical and static permittivity anisotropies of NLC molecules (see Fig. 1).

* Corresponding author. Tel.: +86 2165808341.
E-mail addresses: shzhangguoqing@126.com, zgqw2001@sina.com (G. Zhang), zhonghai.ding@unlv.edu (Z. Ding).

http://dx.doi.org/10.1016/j.nonrwa.2014.08.006
1468-1218/

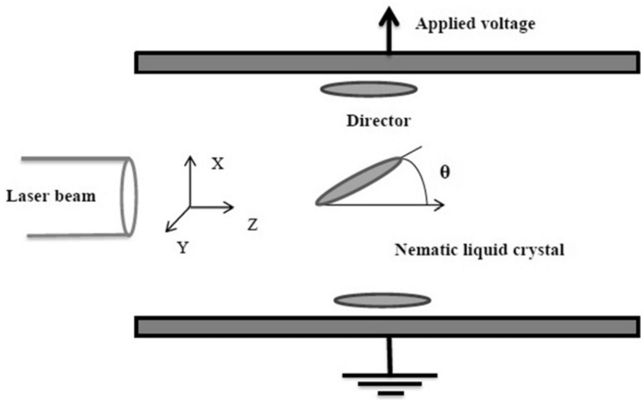

Fig. 1. Sketch of the NLC cell.

The model equations (1.1) and (1.2) provide a very good agreement with experimental data [6,9], and have been studied by several researchers [5,7,11,9]. By using suitable trial function, Minzoni et al. [7] obtained the approximate modulation solutions. Strinic et al. [9] displayed numerically the existence of stable solitons in a narrow threshold region by using a fast Fourier transform algorithm. By using the modified Petviashvili method and the variational method, Aleksic et al. [5] computed the fundamental soliton profiles. Panayotaros and Marchant [11] studied the existence soliton solutions by using a concentration compactness argument, and investigated also their stability properties.

Let $\theta = \theta_0 + \hat{\theta}$, where $\hat{\theta}$ corresponds to the optically induced molecular reorientation. By using the following first-order approximations:

$$\sin^2\theta = \sin^2\theta_0 + \sin(2\theta_0)\hat{\theta} + o(|\hat{\theta}|^2), \tag{1.3}$$

$$\sin(2\theta) = \sin(2\theta_0) + 2\cos(2\theta_0)\hat{\theta} + o(|\hat{\theta}|^2), \tag{1.4}$$

and assuming $\hat{\theta} \ll 1$, one can derive from (1.1) and (1.2) the following lower-order approximation model for the optical field $(E, \hat{\theta})$ [5],

$$2i\frac{\partial E}{\partial z} + \Delta_{x,y}E + \bar{\alpha}\hat{\theta}E = 0, \tag{1.5}$$

$$2\Delta_{x,y}\hat{\theta} + \bar{\beta}\hat{\theta} + \bar{\alpha}|E|^2 = 0, \tag{1.6}$$

where $\bar{\alpha} = \alpha\sin(2\theta_0)$, $\bar{\beta} = 2\beta\cos(2\theta_0)$. By applying the nondimensionalization procedure to (1.5) and (1.6), the following dimensionless dynamical evolution system was derived and studied in [2],

$$2i\frac{\partial E}{\partial z} + \Delta_{x,y}E + \gamma\psi E = 0, \tag{1.7}$$

$$\Delta_{x,y}\psi - c^2\psi + 4\pi|E|^2 = 0, \tag{1.8}$$

where $\gamma \geq 0$ is a parameter, $c \geq 0$ is a constant. Assume a solitary wave of (1.7) and (1.8) is in the form of $E(x, y, z) = \exp(i\omega z)u(x, y)$, where $\omega > 0$ and $u(x, y)$ is a real valued function. Then one obtains from (1.7) and (1.8) the following nonlinear elliptic system,

$$\Delta u - 2\omega u + \gamma u\psi = 0, \quad (x, y) \in \mathbb{R}^2, \tag{1.9}$$

$$-\Delta\psi + c^2\psi = 4\pi u^2, \quad (x, y) \in \mathbb{R}^2. \tag{1.10}$$

The study of existences of solitary wave solutions of (1.7) and (1.8) becomes the study of existences of solutions of the elliptic systems (1.9) and (1.10). One can check easily that if (u, ψ) is a solution of system (1.9) and (1.10), then $u = 0$ if and only if $\psi = 0$. Thus $(u, \psi) = (0, 0)$ is the only trivial solution of system (1.9) and (1.10).

In this paper, we study the existence of nontrivial radially symmetric solutions of the nonlinear elliptic system (1.9) and (1.10). For $p > 1$, let $L^p(\mathbb{R}^2)$ be equipped with the norm

$$\|u\|_p = \left(\int_{\mathbb{R}^2} |u|^p dxdy\right)^{\frac{1}{p}}.$$

Let $H^1(\mathbb{R}^2)$ denote the usual Sobolev space with the scalar product

$$(u, v) = \int_{\mathbb{R}^2} (\nabla u \cdot \nabla v + uv)\, dxdy,$$