Nonlinear Analysis: Real World Applications 22 (2015) 115–128

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Delta wave and vacuum state for generalized Chaplygin gas dynamics system as pressure vanishes ☆

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ARTICLE INFO

Article history:
Received 28 November 2013
Received in revised form 12 July 2014
Accepted 22 August 2014
Available online 15 September 2014

Keywords:
Generalized Chaplygin gas
Riemann problem
Transport equations in zero-pressure flow
Delta wave
Vacuum state

ABSTRACT

This paper is concerned with the Riemann problems for the system of generalized Chaplygin gas dynamics and the formation of Delta wave and vacuum state as pressure vanishes. It is proved that, the limit solutions tend to the two kinds of Riemann solutions to the transport equations in zero-pressure flow, which include a Delta wave formed by a weighted δ-measure and a vacuum state.

1. Introduction

In this paper, we are concerned with the Euler system

$$\begin{cases} \rho_t + (\rho u)_x = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_x = 0, \end{cases} \tag{1}$$

where the variables $\rho(x, t)$ is the density, $u(x, t)$ is the velocity and $p(\rho)$ is the pressure. System (1) with the equation of state

$$p = -A\rho^{-\alpha}, \quad 0 < \alpha < 1, \ A > 0, \tag{2}$$

is called the generalized Chaplygin gas dynamics system.

Guo and Sheng [1] and Wang [2] studied the Riemann problems for the Chaplygin gas and obtained the solutions to the Riemann problems and the interactions of elementary waves. The Riemann solutions to the transport equations in zero-pressure flow in gas dynamics were presented by Sheng and Zhang in [3], in which Delta wave and vacuum state appeared.

The vanishing pressure limit method was investigated first by Li [4] in 2001, in which he obtained the limit of Riemann solutions of system (1) for isothermal gas as pressure vanishes. Then, in 2003 and 2004 Chen and Liu [5,6] obtained the same result and generalized to polytropic gas case. The same problems for relativistic Euler equations were studied by Yin and Sheng [7,8].

In this paper, we focus on the limit of Riemann solutions to the generalized Chaplygin gas dynamics system as pressure vanishes. The organization of the paper is as follows. In Sections 2 and 3, the Riemann problems for the generalized Chaplygin

☆ Sheng and Wang were supported by NSFC 11371240 and 11176015, Shanghai Municipal Education Commission of Scientific Research Innovation Project: 11ZZ84 and the grant of "The First-class Discipline of Universities in Shanghai". Yin was supported by NSFC 11101348.

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http://dx.doi.org/10.1016/j.nonrwa.2014.08.007
1468-1218/

gas dynamics system and the transport equations are analyzed by characteristic analysis. In Section 4, we discuss the limit of Riemann solutions as pressure vanishes.

2. Riemann problems for Euler equations (1)–(2)

In this section, we discuss the Riemann solutions to system (1) with initial data

$$(u,\rho)(x,0)=(u_\pm,\rho_\pm),\quad \pm x>0, \tag{3}$$

where $(u_\pm,\rho_\pm)$ are arbitrary constant states with $\rho_\pm>0$.

For smooth solution, system (1) can be written in the matrix form

$$\begin{pmatrix}\rho\\ u\end{pmatrix}_t+\begin{pmatrix}u & \rho\\ p'(\rho)/\rho & u\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_x=0. \tag{4}$$

It is easy to obtain the eigenvalues

$$\lambda_1=u-\sqrt{A\alpha}\rho^{-\frac{1+\alpha}{2}},\qquad \lambda_2=u+\sqrt{A\alpha}\rho^{-\frac{1+\alpha}{2}}, \tag{5}$$

and the corresponding right-eigenvectors

$$\overrightarrow{r_1}=\left(1,-\frac{c}{\rho}\right)^T,\qquad \overrightarrow{r_2}=\left(1,\frac{c}{\rho}\right)^T, \tag{6}$$

where $c=\sqrt{A\alpha}\rho^{-\frac{1+\alpha}{2}}$. By simple calculation, we get $\nabla\lambda_i\cdot\overrightarrow{r_i}\neq 0,\ i=1,2$, which mean that both the characteristic fields of the generalized Chaplygin gas dynamics system are genuinely nonlinear.

Since system (1) and the initial data (2) remain invariant under the transformation $x\to\alpha x$, $t\to\alpha t$, we seek the self-similar solution $(u,\rho)(x,t)=(u,\rho)(\xi)$ $(\xi=x/t)$ of (1) and (3), in which case the Riemann problem turns into the following boundary-value problem at infinity

$$\begin{cases}-\xi\rho_\xi+(\rho u)_\xi=0,\\ -\xi(\rho u)_\xi+(\rho u^2+p(\rho))_\xi=0,\end{cases} \tag{7}$$

$$(u,\rho)(\pm\infty)=(u_\pm,\rho_\pm).$$

For smooth solutions, (7) can be rewritten as

$$\begin{pmatrix}u-\xi & \rho\\ p'(\rho)/\rho & u-\xi\end{pmatrix}\begin{pmatrix}\rho\\ u\end{pmatrix}_\xi=0, \tag{8}$$

which provides either the general constant solution or the singular solutions called the centered rarefaction waves. More precisely, for a given state (u_-,ρ_-), the possible states (u,ρ) that can be connected to the state (u_-,ρ_-) by a centered backward (forward) rarefaction wave is symbolized by $R_1(u_-,\rho_-)$ $(R_2(u_-,\rho_-))$.

Lemma 1. *The backward and forward rarefaction wave curves are given by*

$$R_1(u_-,\rho_-):\begin{cases}\xi=u-\sqrt{A\alpha}\rho^{-\frac{1+\alpha}{2}},\\ u-\dfrac{2\sqrt{A\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}=u_--\dfrac{2\sqrt{A\alpha}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}},\end{cases}\quad \rho<\rho_-, \tag{9}$$

and

$$R_2(u_-,\rho_-):\begin{cases}\xi=u+\sqrt{A\alpha}\rho^{-\frac{1+\alpha}{2}},\\ u+\dfrac{2\sqrt{A\alpha}}{1+\alpha}\rho^{-\frac{1+\alpha}{2}}=u_-+\dfrac{2\sqrt{A\alpha}}{1+\alpha}\rho_-^{-\frac{1+\alpha}{2}},\end{cases}\quad \rho>\rho_-, \tag{10}$$

respectively. In addition, $\frac{\mathrm{d}u}{\mathrm{d}\rho}<0\ (>0)$ *on* R_1 (R_2).

Similarly, for a given state (u_+,ρ_+), we can obtain $R_1(u_+,\rho_+)$ or $R_2(u_+,\rho_+)$.

For a bounded discontinuous solution, the Rankine–Hugoniot (R–H) relations of (7) are

$$\begin{cases}-\sigma[\rho]+[\rho u]=0,\\ -\sigma[\rho u]+[\rho u^2+p]=0,\end{cases} \tag{11}$$

where $[v]=v_+-v_-,\ v_+=v\,(w+0),\ v_-=v\,(w-0)$ and σ is the velocity of the discontinuity. For a given state (u_-,ρ_-), the possible states (u,ρ) that can be connected to the state (u_-,ρ_-) by a backward or forward shock wave are symbolized by $S_1(u_-,\rho_-)$ or $S_2(u_-,\rho_-)$.