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Space–time regularity of the *mild* solutions to the incompressible generalized Navier–Stokes equations with small rough initial data ☆

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ABSTRACT

In recent paper of Li and Zhai (2010), the authors proved the global-in-time existence and spatial regularity of *mild* solutions to the n-dimensional incompressible generalized Navier–Stokes equations with small initial data u_0 in $Q^{\beta,-1}_{\alpha;\infty}(\mathbb{R}^n) := \nabla \cdot (Q^{\beta}_{\alpha}(\mathbb{R}^n))^n$, $\beta \in (\frac{1}{2}, 1]$ and $\alpha \in [0, \beta)$. In this paper, by using the Fourier localization method, we shall show that the *mild* solution presented by Li and Zhai (2010) satisfies the decay estimates for any space–time derivative involving some borderline Besov space norms. Moreover, the solution has a unique trajectory which is Hölder continuous with respect to space variables.

1. Introduction

In this paper, we consider the following Cauchy problem of n-dimensional incompressible generalized Navier–Stokes (GNS) equations on the half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$, $n \geq 3$:

$$u_t + (-\Delta)^{\beta} u + (u \cdot \nabla)u + \nabla P = 0 \quad \text{in } (x, t) \in \mathbb{R}^{n+1}_+, \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } (x, t) \in \mathbb{R}^{n+1}_+, \tag{1.2}$$

$$u(x, 0) = u_0(x) \quad \text{in } x \in \mathbb{R}^n, \tag{1.3}$$

with $\beta \in (\frac{1}{2}, 1]$, where $u = u(x, t) = (u^1(x, t), \ldots, u^n(x, t))$ and $P = P(x, t)$ stand for, respectively, the fluid velocity and the pressure. The fractional Laplace operator $(-\Delta)^{\beta}$ with respect to space variable x is a Riesz potential operator defined as usual through Fourier transform as $\mathcal{F}((-\Delta)^{\beta} f)(\xi) = |\xi|^{2\beta} \mathcal{F} f(\xi)$, where $\mathcal{F} f(\xi) = \widehat{f}(\xi) = \frac{1}{\sqrt{2\pi}^n} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) \mathrm{d}x$. u_0 is the initial velocity satisfying that $\operatorname{div} u_0 = 0$.

It is well known that by the Duhamel principle, the solution of Cauchy problem (1.1)–(1.3) can be reduced to finding a solution u of the following integral equations:

$$u(t) = e^{-t(-\Delta)^{\beta}} u_0 - \int_0^t e^{-(t-\tau)(-\Delta)^{\beta}} \mathbb{P} \nabla \cdot (u \otimes u)(\tau) \mathrm{d}\tau. \tag{1.4}$$

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Here, $e^{-t(-\Delta)^{\beta}}$ denotes the linear semigroup operator and

$$e^{-t(-\Delta)^{\beta}}f(x) = \mathcal{F}^{-1}(e^{-t|\xi|^{2\beta}}) * f(x).$$

$\mathbb{P} := I + \nabla(-\Delta)^{-1}\mathrm{div}$ is the Helmholtz–Weyl projection operator which has the matrix symbol with components

$$(\widehat{\mathbb{P}}(\xi))_{j,k} = \delta_{jk} - \xi_j\xi_k|\xi|^{-2} \quad \text{with } j, k = 1, 2, \ldots, n,$$

where δ_{jk} is Kronecker symbol, and $\otimes$ denotes tensor product. We notice that the solution of (1.4) is called *mild* solution. We also notice that system (1.1)–(1.3) has the so-called scaling invariance property. More exactly, if $(u(x,t), P(x,t))$ is a solution to system (1.1)–(1.3) with initial data $u_0(x)$, then so is (u_λ, P_λ) for all $\lambda > 0$ with initial data $\lambda^{2\beta-1}u_0(\lambda x)$, where

$$u_\lambda(x,t) = \lambda^{2\beta-1}u(\lambda x, \lambda^{2\beta}t) \quad \text{and} \quad P_\lambda(x,t) = \lambda^{4\beta-2}P(\lambda x, \lambda^{2\beta}t).$$

This scaling invariance is particularly significant for the system and leads to the following definition. A function space is critical for system (1.1)–(1.3) if it is invariant under the scaling $f_\lambda(x) := \lambda^{2\beta-1}f(\lambda x)$.

System (1.1)–(1.3) is a generalization of the usual incompressible Navier–Stokes (NS) equations by replacing the Laplace operator $-\Delta$ in the NS equations by a general fractional Laplace operator $(-\Delta)^{\beta}$ (see Wu [1–4]). In the case when $\beta = 1$, the seminal paper of Leray [5] proved the global existence of finite energy weak solutions to (1.1)–(1.3), but its regularity and uniqueness still remain open. The theory of *mild* solutions to the NS equations is pioneered by Fujita and Kato [6,7], and these works inspired extensive study in the following years on the well-posedness of the NS equations in various critical spaces, see Kato [8], Cannone [9], Koch and Tataru [10], Lemarié-Rieusset [11] and so on. Particularly, Koch and Tataru established the well-posedness for the NS equations with initial data in $BMO^{-1}(\mathbb{R}^n)$. Recently, Xiao in [12,13] generalized the results of Koch and Tataru to a new space $Q^{-1}_{\alpha,\infty}(\mathbb{R}^n)$ for $\alpha \in [0,1)$. For the spatial regularity on the *mild* solution to the NS equations has been studied by many authors, such as Giga and Sawada [14], Sawada [15] and Miura and Sawada [16]. In paper [17], Germain, Pavlović and Staffilani had proved Koch–Tataru's solution u satisfies the following spatial regularity property:

$$t^{\frac{m}{2}}\nabla^m u \in X_{T^*} \quad \text{for all } m \in \mathbb{N},$$

where X_{T^*} is Koch–Tataru's solution existence space. In [18], the authors generalized Germain–Pavlović–Staffilani's results, and obtained that for sufficient small enough $u_0 \in BMO^{-1}$, the global-in-time Koch–Tataru solution satisfies

$$\left\| t^{\frac{m}{2}}\nabla^m u \right\|_{\widetilde{L}^{\infty}(\mathbb{R}_+;\dot{B}^{-1}_{\infty,\infty})\cap\widetilde{L}^{1}(\mathbb{R}_+;\dot{B}^{1}_{\infty,\infty})} \le C\|u_0\|_{BMO^{-1}}(1+\|u_0\|_{BMO^{-1}}) \quad \text{for all } m \in \mathbb{N}.$$

As to the space–time regularity of the *mild* solutions to the NS equations, when $u_0 \in L^n(\mathbb{R}^n)$, in [19], Dong and Du established the result

$$\left\| t^{\frac{m}{2}+k}\partial_t^k\nabla^m u \right\|_{L^{n+2}(\mathbb{R}^n\times(0,T^*))} \le C \quad \text{for all } k, m \in \mathbb{N},$$

where T^* is the maximum existence time. Vary recently, inspired by the results of [19,17], Du in [20] proved that Koch–Tataru's solution is space–time regularity. More precisely, the author established that there holds

$$t^{\frac{m}{2}+k}\partial_t^k\nabla^m u \in X_{T^*} \quad \text{for all } k, m \in \mathbb{N},$$

with the initial data $u_0 \in BMO^{-1}$. On the other hand, with suitable regularities for the solution u to the NS equations, Chemin [21,22] proved that the existence and uniqueness of the trajectory to u, moreover, this trajectory is Hölder continuous with respect to the space variables.

For the general case (1.1)–(1.3), Lions [23] proved that in three space dimensional case, the GNS equations (1.1)–(1.3) possess global classical solutions when $\beta \ge \frac{5}{4}$. Wu [1] obtained a similar result for $\beta \ge \frac{1}{2}+\frac{n}{4}$ in n dimensional case. For the important case $\beta < \frac{1}{2}+\frac{n}{4}$, Wu [2,3] established the local existence and uniqueness results of (1.1)–(1.3) in Besov spaces. Yu and Zhai [24] studied the well-posedness of (1.1)–(1.3) in the largest critical Besov spaces $\dot{B}^{-(2\beta-1)}_{\infty,\infty}(\mathbb{R}^n)$ with $\beta \in (\frac{1}{2}, 1)$. Liu, Zhao and Cui [25] obtained the global existence and stability of *mild* solutions for system (1.1)–(1.3) with small initial data u_0 belonging to the critical pseudomeasure space $PM^a(\mathbb{R}^n)$ (with $a \ge n-(2\beta-1)$ a given parameter, and $\frac{1}{2} < \beta < \frac{n+2}{4}$), where $PM^a(\mathbb{R}^n)$ is defined by

$$PM^a(\mathbb{R}^n) := \left\{ f \in \mathcal{S}' : \widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}^n),\ \|f\|_{PM^a} := \operatorname{ess\,sup}_{\xi\in\mathbb{R}^n}|\xi|^a|\widehat{f}(\xi)| < \infty \right\}.$$

Moreover, they proved that the global-in-time *mild* solution u is spatial analytic, and there holds for $t \in \mathbb{R}_+$,

$$\|\nabla^m u\|_{PM^r} \le Ct^{-\frac{r-a}{2\beta}-\frac{m}{2\beta}} \quad \text{for all } a \le r < n \text{ and } m \in \mathbb{N}.$$

In recent paper [26], inspired by Koch and Tataru [10] and Xiao [12], Li and Zhai proved the well-posedness of the GNS equations (1.1)–(1.3) with initial data in the new critical space $Q^{\beta,-1}_{\alpha,\infty}(\mathbb{R}^n) = \nabla\cdot(Q^{\beta}_{\alpha}(\mathbb{R}^n))^n$ with $\beta \in (\frac{1}{2}, 1]$ and $\alpha \in [0, \beta)$, and they also proved that the *mild* solution is spatial smooth. Here $Q^{\beta}_{\alpha}(\mathbb{R}^n)$ is a generalization of $Q_\alpha(\mathbb{R}^n)$ studied in Essen et al. [27], and Xiao [12,13]. For more results on the regularity of *mild* solutions to (1.1)–(1.3), we refer the readers to Dong and Li [28], Liu Zhao and Cui [29], Wu [4] and Zhou [30].