Nonlinear Analysis: Real World Applications 22 (2015) 423–434

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

On strong solutions to the compressible Hall-magnetohydrodynamic system

Jishan Fan [a], Ahmed Alsaedi [b], Tasawar Hayat [c,b], Gen Nakamura [d], Yong Zhou [e,b,*]

[a] Department of Applied Mathematics, Nanjing Forestry University, Nanjing 210037, PR China
[b] Nonlinear Analysis and Applied Mathematics (NAAM) Research Group, Faculty of Science, King Abdulaziz University, Jeddah 21589, Saudi Arabia
[c] Department of Mathematics, Quaid-i-Azam University, 45320, Islamabad 44000, Pakistan
[d] Department of Mathematics, Inha University, Incheon 402-751, Republic of Korea
[e] School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, PR China

ARTICLE INFO

Article history:
Received 15 April 2014
Accepted 7 October 2014
Available online 3 November 2014

Keywords:
Hall-MHD
Compressible
Strong solutions
Decay

ABSTRACT

In this paper, we consider strong/classical solutions to the 3D compressible Hall-magnetohydrodynamic system. First, we prove the existence of local strong solutions with positive density. Then the existence of global small solutions with small initial data is proved. Optimal time decay rate is also established.

1. Introduction

In this paper, we consider the following compressible Hall-magnetohydrodynamic system [1]:

$$\partial_t \rho + \text{div}\,(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \text{div}\,(\rho u \otimes u) + \nabla p(\rho) - \mu \Delta u - (\lambda + \mu)\nabla \text{div}\, u = \text{curl}\, b \times b, \tag{1.2}$$

$$\partial_t b + \text{curl}\,(b \times u) + \text{curl}\left(\frac{\text{curl}\, b \times b}{\rho}\right) = \Delta b, \tag{1.3}$$

$$\text{div}\, b = 0, \tag{1.4}$$

$$(\rho, u, b)\,|_{t=0} = (\rho_0, u_0, b_0), \tag{1.5}$$

$$\lim_{|x|\to\infty} (\rho, u, b) = (\tilde{\rho}, 0, 0). \tag{1.6}$$

Here ρ is the density of the fluid, u is the fluid velocity field, and b is the magnetic field. The pressure $p(\rho) := a\rho^\gamma$ with positive constants a and $\gamma \geq 1$. λ and μ are two viscosity constants satisfying

$$\mu > 0 \quad \text{and} \quad \lambda + \frac{2}{3}\mu \geq 0.$$

$\tilde{\rho}$ is a positive constant.

* Corresponding author at: School of Mathematics, Shanghai University of Finance and Economics, Shanghai 200433, PR China.
E-mail addresses: fanjishan@njfu.com.cn (J. Fan), nakamuragenn@gmail.com (G. Nakamura), yzhou@mail.shufe.edu.cn (Y. Zhou).

http://dx.doi.org/10.1016/j.nonrwa.2014.10.003
1468-1218/

The applications of the Hall-MHD system cover a very wide range of physical objects, for example, magnetic reconnection in space plasmas, star formation, neutron stars, and geo-dynamo.

When $\rho = 1$, the compressible Hall-MHD system reduces to the incompressible Hall-MHD system, which has received many studies [1–9]. The paper [1] gave a derivation of (1.1)–(1.6) from a two-fluid Euler–Maxwell system. Chae–Degond–Liu [4] proved the local existence of smooth solutions. Chae–Lee [2] and Fan–Ozawa [8] proved some regularity criteria.

When the Hall effect term $\operatorname{curl}\left(\frac{\operatorname{curl} b\times b}{\rho}\right)$ is neglected, the system (1.1)–(1.6) reduces to the well-known compressible isentropic MHD system, which has received many studies [10–19]. The local strong solution was proved by Fan–Yu [10]. Hu–Wang [11,12], Fan–Yu [13], and Ducomet–Feireisl [14] established the global weak solutions. The low Mach number limit problem was studied by Hu–Wang [15] and Jiang–Ju–Li [16]. Li–Yu [17], Chen–Tan [18], and Pu–Guo [19] showed the time decay of smooth solutions.

In this paper, we will prove the existence of local strong solutions to (1.1)–(1.6) with positive density in Section 2. Then the existence of global small solutions and the optimal time decay rate of classical solutions with small initial data will be shown in Section 3.

2. Local existence of strong solutions

The main theorem in this section reads

Theorem 2.1. *Let $0 < \frac{1}{C} \le \rho_0, \tilde\rho \le C$, $\nabla\rho_0 \in H^1$ and $u_0, b_0 \in H^2$ with $\operatorname{div} b_0 = 0$ in $\mathbb{R}^3$. Then the problem (1.1)–(1.6) has a unique strong solution (ρ, u, b) satisfying*

$$\begin{aligned} &\frac{1}{C} \le \rho \le C,\ \nabla\rho,\ \rho_t \in L^\infty(0,T;H^1),\\ &u, b \in L^\infty(0,T;H^2)\cap L^2(0,T;H^3),\\ &u_t, b_t \in L^\infty(0,T;L^2)\cap L^2(0,T;H^1)\end{aligned} \tag{2.1}$$

for some $0 < T \le 1$.

We will prove Theorem 2.1 by Banach fixed point theorem. We denote the Banach space

$$\mathcal{A} := \{\tilde u \in \mathcal{A} \mid \|\tilde u\|_{\mathcal{A}} \le A\}$$

with the norm

$$\|\tilde u\|_{\mathcal{A}} := \|\tilde u\|_{L^\infty(0,T;H^2)} + \|\tilde u\|_{L^2(0,T;H^3)} + \|\partial_t\tilde u\|_{L^\infty(0,T;L^2)} + \|\partial_t\tilde u\|_{L^2(0,T;H^1)}. \tag{2.2}$$

Let $\tilde u, \tilde b \in \mathcal{A}$ be given, we consider the following linear problem:

$$\partial_t\rho + \operatorname{div}(\rho\tilde u) = 0, \qquad \lim_{|x|\to\infty} = \tilde\rho, \tag{2.3}$$

$$\rho(\cdot,0) = \rho_0, \tag{2.4}$$

$$\partial_t b + \operatorname{curl}(b\times\tilde u) + \operatorname{curl}\left(\frac{\operatorname{curl} b\times\tilde b}{\rho}\right) = \Delta b, \qquad \operatorname{div} b = 0, \tag{2.5}$$

$$b(\cdot,0) = b_0, \qquad \lim_{|x|\to\infty} b = 0, \tag{2.6}$$

$$\rho\partial_t u + \rho\tilde u\cdot\nabla u + \nabla p(\rho) - \mu\Delta u - (\lambda+\mu)\nabla\operatorname{div} u = \operatorname{curl} b\times b, \tag{2.7}$$

$$u(\cdot,0) = u_0, \qquad \lim_{|x|\to\infty} u = 0. \tag{2.8}$$

Let (u, b) be the unique strong solution to the above problem, we define the fixed point map $F : (\tilde u, \tilde b) \in \mathcal{A}\times\mathcal{A} \to (u, b) \in \mathcal{A}\times\mathcal{A}$ with $\tilde u(\cdot,0) = u_0$, $\tilde b(\cdot,0) = b_0$, $\operatorname{div}\tilde b = 0$, $\lim_{|x|\to\infty}(\tilde u,\tilde b) = (0,0)$. We will prove that the map F mapping $\mathcal{A}\times\mathcal{A}$ into $\mathcal{A}\times\mathcal{A}$ for suitable constant A and small T and F is a contraction mapping on $\mathcal{A}\times\mathcal{A}$ and thus F has a unique fixed point in $\mathcal{A}\times\mathcal{A}$. This proves the result.

First, we have the following lemma

Lemma 2.2. *Let $\tilde u \in \mathcal{A}$ be given. Then the problem (2.3) and (2.4) has a unique solution ρ satisfying*

$$\frac{1}{C} \le \rho \le C, \qquad \|\nabla\rho\|_{L^\infty(0,T;H^1)} \le C, \qquad \|\rho_t\|_{L^\infty(0,T;H^1)} \le CA \tag{2.9}$$

for some small $0 < T \le 1$.