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Analyticity of the streamlines and of the free surface for periodic equatorial gravity water flows with vorticity

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ABSTRACT

We prove analyticity of the streamlines beneath the surface and the smoothness of the free surface for geophysical equatorial water flows with a general Hölder continuously differentiable underlying vorticity distribution under the assumption of no stagnation points in the flow. Moreover, we prove that the real-analyticity of the vorticity function implies the real-analyticity of the free surface.

1. Introduction

The study of periodic traveling water waves arising as the free surface of an irrotational flow with a flat bed originates at the beginning of the 19th century, being confined first to the investigation of waves of small amplitude, which were satisfactorily approximated by sinusoidal curves within the linear theory. To describe waves that are flatter near the trough and have steeper elevations near the crest a nonlinear approach was required and was in fact undertaken, the first rigorous results concerning the existence of wave trains in irrotational flows occurring in the last decades, see for instance the case of the Stokes waves [1] and the flow beneath them (particle trajectories, behavior of the pressure) cf. [2–5].

A step further in the development of the water wave theory was the realization that wave current interactions necessitate the incorporation of vorticity into the problem, cf. [6–8]. Nevertheless, the difficulties generated by the presence of vorticity have prevented a rigorous mathematical development, which appeared only relatively recently in [9], where the existence of small and large amplitude steady periodic gravity water waves with a general (continuous) vorticity distribution was proved. The paper [9] was followed by a series of mathematically rigorous results concerning symmetry [10–12], regularity of the free surface and of the stream lines [13–17], allowing for stagnation points and critical layers [18–21] or presenting a discontinuous (piecewise constant) [22], merely bounded [23–25] or even unbounded vorticity distribution [26].

An important aspect in the study of water flows, that was recently taken into account, is the Earth's rotation, a feature which is relevant in the study of large scale physical phenomena through the presence of the Coriolis force in the governing equations, cf. [27–29]. Of particular interest are the geophysical processes that occur in the equatorial region. One of the reasons is that the Equator has the peculiar property of behaving as a natural wave guide, i.e. equatorially trapped zonal waves decay exponentially away from the Equator in oceans [30–36].

One instance of these large scale phenomena which occur in the equatorial region is the Equatorial Undercurrent (EUC) which stretches throughout the extent of the Pacific Ocean (over 13,000 km), being confined to a shallow layer centered on the Equator which is less than 200 m deep and is typically about 300 km in width. Due to the prevailing westward direction of equatorial winds, the wind-generated surface wave of EUC propagates westwards while at depths of several tens of meters the flow reverses, cf. [37].

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Given the smallness in width of EUC, we can disregard the variations of the Coriolis parameter and use the f-plane approximation in the equatorial region, cf. [31,38].

The existence of equatorial waves (without stagnation points) was proved in [32] for the case of constant vorticity, the case of a general regular vorticity being resolved in [39]. It is worth mentioning here that an explicit solution for the geophysical (edge) wave problem in the f-plane approximation was given in [40]. Another recent result pertaining to geophysical flows concerns the symmetry of steady equatorial wind waves which was proved in [41]. The existence of (capillary)-gravity equatorial water waves with constant vorticity and allowing for stagnation points and overhanging profiles was proved in [42], where also a regularity result for the free surface was established.

We are concerned in this paper with proving that a regularity result of the streamlines together with the analyticity of the free surface, obtained in [13], in the case of gravity waves, can be extended to the geophysical setting of equatorial water waves.

Let us briefly summarize the content of this paper. In Section 2 we recall the governing equations for the f-plane approximation of geophysical flows, cf. [31,32]. The main results of the paper, Theorems 3.1 and 3.3, together with their proofs are given in Section 3.

2. The governing equations

We consider a reference frame having the origin located at a point on the Earth's surface and which rotates with the Earth with the x-axis chosen horizontally due east, the y-axis horizontally due north and the z-axis pointing vertically upwards. We set $z = -d$ to be the lower boundary of the layer to which the effect of the wind is confined and $z = \eta(x, y, t)$ be its upper free boundary. When considering the f-plane approximation to the full geophysical equations, which is valid near the latitudinal strip near the Equator [27], the governing equations for the region $-d \le z \le \eta(x, y, t)$ are, cf. [27,29], the Euler equations

$$\begin{cases} u_t + uu_x + vu_y + wu_z + 2\omega w = -\dfrac{1}{\rho}P_x, \\ v_t + uv_x + vv_y + wv_z = -\dfrac{1}{\rho}P_y, \\ w_t + uw_x + vw_y + ww_z - 2\omega u = -\dfrac{1}{\rho}P_z - g, \end{cases} \tag{2.1a}$$

where (u, v, w) denotes the velocity field of the fluid, $\omega = 73 \cdot 10^{-6}$ rad/s is the (constant) rotational speed of the Earth[1] around the polar axis towards the east, P is the pressure, g is the (constant) gravitational acceleration at the Earth's surface and ρ is the constant fluid density, an assumption which, according to [28], is appropriate for this layer and implies the equation of mass conservation

$$u_x + v_y + w_z = 0. \tag{2.1b}$$

Due to the Equator acting as a wave guide we may approximate the flows as being two-dimensional, independent of the y-coordinate and with $v \equiv 0$. Moreover, we are seeking periodic steady traveling waves, which means that the velocity field, the pressure and the free surface are periodic in the x-direction and are exhibiting an (x, t)-dependence of the form $(x - ct)$ with $c < 0$, where $|c|$ is the westward propagation speed of the surface wave. The equations of motion are supplemented by the kinematic boundary conditions

$$w = (u - c)\eta_x \quad \text{on } z = \eta(x), \tag{2.1c}$$

and

$$w = 0 \quad \text{on } z = -d, \tag{2.1d}$$

expressing the impermeability of the free surface and of the bottom layer $z = -d$,[2] and the dynamic boundary condition

$$P = P_{\text{atm}} \quad \text{on } z = \eta(x - ct) \tag{2.1e}$$

which decouples the motion of the water from the air above it—cf. the discussion in [6]. We denote by P_{atm} the constant atmospheric pressure.

Due to the vanishing of v and the independence of the y-coordinate we can identify the vorticity $\gamma = (0, u_z - w_x, 0)$ with

$$\gamma = u_z - w_x. \tag{2.1f}$$

1 Taken to be a perfect sphere of radius 6371 km.

2 Due to the fact that the effect of the wind is confined to the region bounded by the free surface and the line $z = -d$, the latter can be thought of as the bed of the flow.