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Reaction terms avoiding aggregation in slow fluids

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A B S T R A C T

We propose a new mathematical model describing the fertilization process of corals and some other invertebrates. As a result we obtain a Keller–Segel system with reaction terms coupled with a Stokes equation. We prove in dimension two that independently of the size of $|\rho_0|_{L^1}$, there exist global weak solutions. Finally we discuss some open problems arising from our research.

1. Introduction

In [1,2] the authors consider the effect chemotactic attraction may have on reproduction of some invertebrates, such as sea urchins, anemones and corals. In particular they consider the phenomenon of broadcast spawning whereby males and females release sperm and egg gametes into the surrounding flow. For the coral spawning problem, there is experimental evidence that eggs release a chemical that attracts sperm, see [3–6]. Thus we have a phenomenon that leads us to consider a mathematical Keller–Segel model for chemotaxis that should include the effect of the surrounding fluid. In [1] the following model was proposed in order to analyze this fertilization process:

$$\rho_t + \mathbf{u}\cdot\nabla\rho = \Delta\rho + \chi\nabla\cdot\left(\rho\nabla(\Delta)^{-1}\rho\right) - \varepsilon\rho^q, \quad \rho(x,0),\ x\in\mathbb{R}^n. \tag{1}$$

Here ρ denotes the unknown population density (it is assumed that densities of spermatozoids and egg gametes are identical), meanwhile the vector field $\mathbf{u}$, modeling the ambient ocean flow, is considered to be a given function which is divergence-free. The term $\chi\nabla\cdot(\rho\nabla(\Delta)^{-1}\rho)$ with $\chi > 0$ models the standard chemotactic phenomena, and the last term $-\varepsilon\rho^q$ models the reaction (fertilization). The parameter ε regulates the strength of the fertilization process. Enough conditions for global existence in time are given in [1, Th. 4.1] for the case that q is an integer larger than two, meanwhile the critical reaction case $q = 2$ is approached in [2, Th. 3.1]. On the other hand, the asymptotic behavior of the quantity

$$m_0(t) = \int_{\mathbb{R}^n} \rho(x,t)dx,$$

which represents the total fraction of the unfertilized eggs by time t, is also researched.

Some other models describing different phenomena of chemoattraction in fluids have recently been considered in the mathematical literature, see for example [7–11]. The aim of this paper is to improve the model (1) for the case $q = 2$, which

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corresponds to the biologically meaningful case, by assuming that the fluid velocity is unknown and modeling it through a Stokes equation. The use of the Stokes equations instead of the Navier–Stokes equation is justified if we assume that the fluid flow is slow. On the other hand we want to take into account that the chemical is also transported by the fluid, which again contrasts with (1). Then we obtain the following Keller–Segel–Stokes model

$$c_t + \mathbf{u}\cdot\nabla c = \Delta c - a_1 c + \rho, \tag{2}$$

$$\rho_t + \mathbf{u}\cdot\nabla\rho = \Delta\rho - \chi\nabla\cdot(\rho\nabla c) - \varepsilon\rho^2, \tag{3}$$

$$a_2\mathbf{u}_t + \nabla P = \eta\Delta\mathbf{u} - \rho\nabla\phi, \tag{4}$$

$$\nabla\cdot\mathbf{u} = 0. \tag{5}$$

In our model the unknowns are the chemical concentration c, the cell density ρ, the fluid velocity $\mathbf{u}$ described by the Stokes equations and the pressure P of the fluid. The parameters a_1, a_2, χ, η and ε are nonnegative constants. The coupling of chemotaxis and fluid is realized through the terms $u\cdot\nabla\rho$ modeling the transport of cells, $u\cdot\nabla c$ modeling the transport of chemical substances, and the external force $-\rho\nabla\phi$ exerted on the fluid by cells. One example of potential function $\phi = \phi(x,t)$ arises when considering gravity force. Then we can take $\phi = kx_1$ for a constant $k \in \mathbb{R}$ depending on fluid mass density, cell mass density and gravity acceleration. Another possibility is the existence of a centrifugal force ϕ. We are mainly interested in studying our model in dimension $n = 2$ and $n = 3$ corresponding to the cases of physical interest. In this paper we approach the two dimensional case. In the case of coral reproduction, dispersion phenomena could play an important role in the biological description of the whole process and therefore an unbounded domain would be suitable for our model. However, we also consider that for other approaches a bounded smooth domain $\Omega \subset \mathbb{R}^2$ could be well taken into consideration. More precisely we will assume throughout the whole paper that our model is defined in $\Omega \subset \mathbb{R}^2$ satisfying either $\Omega = \mathbb{R}^2$, or else Ω representing a bounded smooth domain.

For the cell density ρ and the chemical concentration c we assume, in the case of having an smooth bounded domain, homogeneous Neumann conditions, i.e.,

$$\frac{\partial\rho}{\partial n} = \frac{\partial c}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{6}$$

We take nonnegative initial data,

$$\rho(x,0) = \rho_0(x) \ge 0, \quad x \in \Omega, \tag{7}$$

$$c(x,0) = c_0(x) \ge 0, \quad x \in \Omega. \tag{8}$$

For the fluid velocity we assume no-slip boundary condition

$$\mathbf{u} = 0 \quad \text{on } \partial\Omega. \tag{9}$$

We assume

$$\nabla\phi \in L^\infty(\Omega) \tag{10}$$

to be known.

In the next section we introduce the tools and the notation of the Stokes equations theory needed in this paper. Then we give a formal definition of weak solution for system (2)–(5). Next, in the third section we prove local existence using the Schauder fixed point theorem. Then we show how to find the corresponding estimates that allow us to conclude global existence. Finally in the last section we discuss some open problems arising from our research.

2. Definition of weak solution

Taking into account that we are working with a Stokes equation, we want to introduce first of all the corresponding theory along with the notation that we will use through this paper. Let Ω be a subset in $\mathbb{R}^2$ satisfying

$$\Omega = \mathbb{R}^2 \text{ or else } \Omega \subset \mathbb{R}^2 \text{ is a bounded smooth domain.} \tag{11}$$

Let us define

$$L^q_\sigma(\Omega) := \overline{\left\{v \in C_0^\infty(\Omega)^n : \operatorname{div} v = 0\right\}}^{L^q(\Omega)^n}.$$

It is known that every function $f \in L^q(\Omega)^n$ can be uniquely decomposed as

$$f = f_0 + \nabla Q, \quad \text{(Helmholtz decomposition)}$$

with $f_0 \in L^q_\sigma(\Omega)$, $Q \in L^q_{loc}(\Omega)$, $\nabla Q \in (L^q(\Omega))^n$ and

$$\|\nabla Q\|_q \le C\,\|f\|_q \quad \text{and} \quad \|Q\|_{L^q(\Omega\cap B_0)} \le C\,\|Q\|_q\,,$$