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A note on gradient-type systems on fractals

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HIGHLIGHTS

- We apply a Ricceri-type critical-point theorem to elliptic systems on fractals.
- We emphasize the special structure of the Sierpinski gasket.
- We overcame difficulties arising from the non-smooth structure of fractals.

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ABSTRACT

The purpose of the paper is to point out that well-established methods to study the multiplicity of solutions of nonlinear elliptic problems defined on open sets in Euclidean spaces can be also used in the case of problems defined on fractals. More exactly, using variational methods and an abstract critical-point theorem by B. Ricceri, we prove the existence of three nonzero weak solutions of certain gradient-type systems defined on a famous fractal, the Sierpinski gasket. The paper emphasizes the way we overcame the difficulties arising from the major structural differences between the highly non-smooth Sierpinski gasket and the open subsets of Euclidean spaces on which gradient-type systems are usually considered.

1. Introduction

The origins of *analysis on fractals* lie in B.B. Mandelbrot's book [1] where fractals are proposed as models for different physical phenomena. Subsequently, the Laplacian on fractals, which first appeared in physics as a tool for investigating the percolation effect and various transport processes (in classical as well as in quantum mechanics), became the subject of intensive mathematical research. An overview of these researches can be found, for instance, in the introduction of R.S. Strichartz's book [2]. Here we only point out that defining the Laplacian on a general fractal implies to cope with considerable difficulties and that, over the years, several definitions have been proposed that are applicable to certain classes of fractals. For example, in the construction that goes back to J. Kigami (e.g., [3–6]) the Laplacian is defined as the limit of discrete differences on graphs approximating the fractal, a method that fits with so-called *post-critically-finite fractals*. Another approach was taken by U. Mosco (e.g., [7–9]), who introduced a framework for the Laplacian by taking as a starting point a Dirichlet form that reflects the self-similarities of the underlying fractal. This framework led to the very general theory of *variational fractals*.

Once a Laplacian has been defined on a fractal, one begins to study elliptic (linear and nonlinear) problems on it. In the last twenty years there have been many contributions to this area. The papers [10–19] are only a few examples in this sense.

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This paper is devoted to nonlinear elliptic systems on a special fractal set. It is worthy of mention that nonlinear elliptic systems, defined on (bounded or unbounded) open sets in $\mathbb{R}^n$, have been intensively studied in the last three decades (see, for instance, the references in [20]). An important class of such systems is represented by those of gradient-type. In this sense, we point to Chapter 4 of [21] and to the references mentioned there.

Recently, in [22], the study (by means of variational methods) of gradient-type systems defined on fractals has been initiated. More exactly, using a variational principle that goes back to B. Ricceri (see [23]), the authors prove in [22] the existence of a sequence of weak solutions for parametric quasilinear gradient-type systems defined on the Sierpinski gasket. We point out that the Sierpinski gasket is of particular interest in fractal theory since it is a typical example of a post-critically-finite fractal.

In the present paper, we propose another variational method to prove the existence of multiple (finitely many) weak solutions of gradient-type systems defined on the Sierpinski gasket: using an abstract four-critical-point theorem stated by B. Ricceri in [24], we show that certain one-parameter elliptic gradient-type systems defined on the Sierpinski gasket have at least three nonzero weak solutions. As far as we know, this is the first application of a finitely many-critical-point theorem of Ricceri type to elliptic systems defined on fractals. The major difficulty we had to cope with is related to a specific feature of the four-critical-point theorem: one has to show the existence of a function with certain properties which belongs to that space of real-valued functions defined on the Sierpinski gasket that corresponds to the Sobolev spaces in case of classical (i.e., open) subsets of $\mathbb{R}^n$. By contrast with the application of Ricceri's abstract four-critical-point theorem to elliptic problems defined on classical subsets of $\mathbb{R}^n$, where this function can be constructed explicitly (e.g., in the proof of Theorem 2 in [24] this function, denoted by u_2, is actually a constant function, and in the proof of Theorem 1.1 in [25] this function, denoted by u_c, is constructed with the aid of the distance function from a point to a certain compact subset of the open, bounded, and connected subset of $\mathbb{R}^n$ where the elliptic problem is considered), in our application of Ricceri's theorem to the fractal case we have to involve Urysohn's Lemma to prove the existence of such a function. Nevertheless, one can give concrete examples that allow an explicit construction of such functions, using a specific method that characterizes the Sierpinski gasket (see (1) in Example 4.1).

Understanding and dealing with the phenomena in the case of gradient-type systems on the Sierpinski gasket is the first step for the study of these systems in the more general setting of post-critically-finite fractals.

2. Preliminaries

Notations. (1) We denote by $\mathbb{N}$ the set of natural numbers $\{0, 1, 2, \ldots\}$, by $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ the set of positive naturals, and by $|\cdot|$ the Euclidean norm on the spaces $\mathbb{R}^n$, $n \in \mathbb{N}^*$. Throughout the paper, the spaces $\mathbb{R}^n$ are considered to be endowed with the Euclidean topology.

(2) If X is a set, Y a nonempty subset of X, and $f: X \to \mathbb{R}$ a real-valued function, then we will use the notations

$$\inf_Y f := \inf\{f(y) \mid y \in Y\} \quad \text{and} \quad \sup_Y f := \sup\{f(y) \mid y \in Y\}.$$

(3) If X is a real normed space, then X^* stands for its dual space.

Our study of gradient-type systems on the Sierpinski gasket is based on the following abstract four-critical-point theorem by B. Ricceri which is a consequence of a more general result (see Theorem 1 in [24]).

Theorem 2.1. *Let X be a reflexive real Banach space, let Φ, Ψ, $J: X \to \mathbb{R}$ be functionals, let $z_0, z_1 \in X$, and let $\rho > 0$ be a real number such that the following conditions hold:*

(i) *Φ is a coercive, sequentially weakly lower semicontinuous C^1-functional whose derivative admits a continuous inverse on X^*.*
(ii) *Ψ and J are C^1-functionals with compact derivatives.*
(iii) *z_0 is a strict local minimum of the functional Φ and $\Phi(z_0) = \Psi(z_0) = J(z_0) = 0$.*
(iv) *The inequalities*

$$\max\left\{\limsup_{u\to z_0}\frac{J(u)}{\Phi(u)},\ \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)}\right\} \le 0 \tag{1}$$

and

$$\max\left\{\limsup_{u\to z_0}\frac{\Psi(u)}{\Phi(u)},\ \limsup_{\|u\|\to\infty}\frac{\Psi(u)}{\Phi(u)}\right\} < 1 \tag{2}$$

hold.
(v) *$0 < J(z_1) = \sup_{(\Phi-\Psi)^{-1}(]-\infty,\rho])} J < \sup_X J$ and $\Phi(z_1) \le \Psi(z_1)$.*

Then there exists $\lambda^* > 0$ such that the functional $\Phi - \lambda^* J - \Psi$ has at least four critical points, z_0 being one of them. Moreover, two of these four critical points (different from z_0) are actually global minima of $\Phi - \lambda^* J - \Psi$.