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The existence of multiple equilibrium points in a global attractor for some symmetric dynamical systems

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ABSTRACT

In this paper, we are mainly concerned with some properties of the global attractor for some symmetric dynamical systems with a Lyapunov function in a Banach space. Under some suitable assumptions, we give an estimate of lower bound of Z_2 index of the global attractor and get the existence of the multiple equilibrium points in the global attractor for the symmetric dynamical systems. As an application of these results, we consider the existence of multiple stationary solutions for some semilinear reaction–diffusion equations.

1. Introduction

The understanding of the asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. The main method to treat this problem for a dissipative system is to consider the existence of a global attractor and analyze the structure of the global attractor, which is an invariant compact set and attracts all bounded subsets in some space. In general, the global attractor has a very complicated topological structure which represents the complexity of the long-time behavior of the system (see [1–4]).

It is well-known (see [4]) that if a system or a semigroup $\{S(t)\}_{t\geq 0}$ has a global attractor $\mathcal{A}$ in a Banach space X and has a Lyapunov function F on some neighborhood of $\mathcal{A}$, then

$$\mathcal{A} = W^u(\mathcal{E}),$$

where $\mathcal{E}$ is the set of all fixed points and $W^u(\mathcal{E})$ is the unstable manifold of $\mathcal{E}$. The Lyapunov function F on some neighborhood $\mathcal{N}$ of $\mathcal{A}$ (see [4]) means that for each $u_0 \in \mathcal{N}$, the function $t \to F(S(t)u_0)$ is decreasing, along with if $F(S(\tau)u) = F(u)$ for some $\tau > 0$, then u is a fixed point of the semigroup $\{S(t)\}_{t\geq 0}$. Furthermore, if $\mathcal{E}$ is discrete, then

$$\mathcal{A} = W^u(\mathcal{E}) = \bigcup_{z\in\mathcal{E}} W^u(z).$$

On the other hand, we also note that

$$\mathcal{A} = \{\theta(t) : \theta(t) \text{ is a complete bounded orbit of } \{S(t)\}_{t\geq 0}\}.$$

In this case, each complete bounded orbit $\theta(t)$ is always connected to some pair of fixed points of the semigroup $\{S(t)\}_{t\geq 0}$, and $\theta(t)$ is contained in the unstable manifold from the one point and the stable manifold from the other point. Thus, the

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structure of an attractor can be fully specified by the set of stationary points which are joined to each other. From this point of view, we know that if the number of equilibrium points in $\mathcal{A}$ is very large, the topology of the global attractor is sufficiently complicated. Therefore, it is meaningful for us to explore the multiplicity of equilibrium points in the global attractor $\mathcal{A}$.

Recently, some authors have considered the dimension of attractors for some parabolic equations. For example, in [5], the authors proved the fractal dimension of the global attractor for some p-Laplacian equations is infinite by use of Z_2 index. The infinite-dimension of attractors for evolution equations with p-Laplacian and their Kolmogorov entropy was proved in [6]. In [7], the authors have proved the finite and infinite dimension of attractors for porous media equations. The finite-dimension of attractors and exponential attractors for degenerate doubly nonlinear equations was proved in [8]. In [9], the authors have proved the infinite-dimension of attractors for parabolic equations with p-Laplacian in heterogeneous medium.

In [10–12], the authors considered the bifurcation problem for the Chaffee–Infante equation

$$\begin{cases} u_t - u_{xx} = \lambda(u - u^3), \\ u(0) = u(\pi) = 0 \end{cases}$$

with $\lambda \geq 0$ and proved there exists $2n+1$ fixed points in the global attractor for $n^2 < \lambda < (n+1)^2$. From their analysis, we know that the origin is stable for $0 \leq \lambda < 1$, but there will appear another pair unstable fixed points once $\sqrt{\lambda}$ passes through each eigenvalue (i.e., each integer value). Therefore, there will be successive saddle node bifurcations from the origin that produce all pairs of fixed points for λ large enough.

On the other hand, we notice that the authors used Z_2 index to prove the number of the solutions of the equation $-\Delta u = \lambda f(u)$ tends to infinity with $\lambda \to \infty$ under the assumptions that f is subcritical and odd in [13]. In [14], the authors considered the multiplicity of solutions for the perturbation problem $-\Delta u = f(x, u) + \epsilon g(x, u)$ and used Z_2 index to prove for any natural number j, there exists some $\epsilon_j > 0$ such that there exist at least j distinct solutions for any $0 < \epsilon \leq \epsilon_j$ under some suitable conditions. The existence of a sequence of distinct solutions for $-\Delta u = \lambda f(x, u)$ under the assumptions that f is subcritical and odd and the similar results with the one in [14] was proved in [15].

Furthermore, we also notice that the cubic–quintic nonlinear Schrödinger equations (see [16–18])

$$i\frac{\partial u}{\partial t} + \Delta u = \alpha_1 u - \alpha_3 |u|^2 u + \alpha_5 |u|^4 u$$

extensively appears in a great variety of physical problems, where $|u|^2u$ and $|u|^4u$ is the self-focusing and self-defocusing term in the cubic–quintic nonlinear Schrödinger equations, respectively (see [16,17]). It is a model in superfluidity [19], descriptions of bosons [20] and of defectons [21], the theory of ferromagnetic and molecular chains [22,23], and in nuclear hydrodynamics [24].

Motivated by the above papers, in this paper, we consider the existence of multiple equilibrium points in a global attractor for some symmetric dynamical systems and obtain some similar results as in [13,14]. We first combine Z_2 index (see [25,5]) with the idea of invariant sets of semi-flows originated from [26] to estimate the lower bound of Z_2 index of two disjoint subsets of the global attractor for which one subset is located in the area where the Lyapunov function F is positive and the other subset is located in the area where the Lyapunov function F is negative, then we use this result to obtain the existence of multiple equilibrium points in a global attractor for some symmetric dynamical systems with a Lyapunov function F. As an application of these results, in Section 4, we will consider more general equations than the following equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta u - \alpha_3 |u|^{r-2}u + \alpha_5 |u|^{p-2}u = 0, & (x, t) \in \Omega \times \mathbb{R}^+, \\ u = 0, & (x, t) \in \partial\Omega \times \mathbb{R}^+, \\ u(x, 0) = u_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^d$, α_3, α_5 are two positive parameters, $p > r > 2$ are constants. In case $r = 4$, $p = 6$, the nonlinearities $|u|^2u$ and $|u|^4u$ of (1.1) is the same as the self-focusing and self-defocusing term of the cubic–quintic nonlinear Schrödinger equations.

This paper is organized as follows: In Section 2, for the convenience of the readers, we review briefly some basic concepts and give some lemmas. In Section 3, we prove the existence of multiple equilibrium points in a global attractor for some symmetric dynamical systems with a Lyapunov function by combining Z_2 index with the idea of invariant sets of semi-flows. As an application of these results, the existence of multiple stationary solutions for some symmetric reaction–diffusion equations under some suitable assumptions is considered in Section 4.

Throughout this paper, let X be a Banach space endowed with the norm $\|\cdot\|_X$, and let C be a positive constant, which may be different from line to line (and even in the same line).

2. Some preliminaries

In this section, we review briefly some basic concepts and give some lemmas, which can be referred to [27,28,25].

Denote the class of closed symmetric subsets of X by

$$\mathbb{A} = \{A \subset X : A \text{ is closed}, A = -A\}.$$