Nonlinear Analysis: Real World Applications 19 (2014) 54–66

Contents lists available at ScienceDirect

Nonlinear Analysis: Real World Applications

journal homepage: www.elsevier.com/locate/nonrwa

Asymptotic behavior of trembling fluids

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ARTICLE INFO

Article history:
Received 20 July 2012
Accepted 23 February 2014

ABSTRACT

A general problem modeling trembling fluids with applications in electro, magneto or thermo-rheological fluids is consider in this work. The fluid flow is governed by the generalized Navier–Stokes equations with a variable q-structure. We prove that the solutions of the associated initial and boundary-value problem extinct in a finite time as long as the trembling fluid remains in the pseudo-plastic zone. For trembling fluids strictly confined to the dilatant zone or that can cross the Newtonian barrier and eventually go back, we study the large time behavior of the solutions. Perturbations of the asymptotically stable equilibrium are analyzed as well.

1. Introduction

Let us consider a general cylinder

$$Q_T := \Omega \times [0,T], \quad \text{with } \Gamma_T := \partial\Omega \times [0,T],$$

where $\Omega \subset \mathbb{R}^N$, $N \geq 2$, is a bounded domain with a compact boundary $\partial\Omega$, and $0 < T < \infty$. In this cylinder, we consider the generalized Navier–Stokes equations:

$$\operatorname{div} \mathbf{u} = 0 \quad \text{in } Q_T, \tag{1.1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = \mathbf{f} - \nabla p + \mathbf{div}\,\mathbf{S} \quad \text{in } Q_T, \tag{1.2}$$

supplemented with the following initial and boundary conditions

$$\mathbf{u} = \mathbf{u}_0 \quad \text{in } \Omega \text{ for } t = 0, \tag{1.3}$$

$$\mathbf{u} = \mathbf{0} \quad \text{on } \Gamma_T. \tag{1.4}$$

Here, $\mathbf{u}$ is the velocity field, p stands for the pressure divided by the constant density and $\mathbf{f}$ is the external forces field. $\mathbf{S}$ is the deviatoric part of the Cauchy stress tensor, which depends on $(\mathbf{x}, t)$ and on the strain rate tensor $\mathbf{D}$:

$$\mathbf{S} = \nu(D_{II})\mathbf{D}, \qquad \mathbf{D} \equiv \mathbf{D}(\mathbf{u}) = \frac{1}{2}\left(\nabla\,\mathbf{u} + \nabla\,\mathbf{u}^T\right), \qquad D_{II} = \frac{1}{2}|\mathbf{D}|^2, \tag{1.5}$$

where $|\mathbf{D}|$ accounts for the shear rate and D_{II} is the second invariant of $\mathbf{D}$. We prescribe the velocity $\mathbf{u}_0$ at the initial time and we assume the fluid adheres to the fixed boundary for all time. Many constitutive laws have been proposed in the rheological literature to model diverse non-Newtonian fluids in different flow conditions (see *e.g.* Barnes et al. [1]), but here we consider

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http://dx.doi.org/10.1016/j.nonrwa.2014.02.005
1468-1218/

a general deviatoric stress tensor $\mathbf{S}$ having a variable q-structure in the following sense:

(A) $\mathbf{S}: Q_T \times \mathbb{M}^N_{\rm sym} \to \mathbb{M}^N_{\rm sym}$ is a Charathéodory function;
(B) growth condition: $|\mathbf{S}(\mathbf{x},t,\mathbf{A})| \leq C|\mathbf{A}|^{q(\mathbf{x},t)-1}$ for all $\mathbf{A}$ in $\mathbb{M}^N_{\rm sym}$ and for a.a. $(\mathbf{x},t)$ in Q_T;
(C) coercivity condition: $\mathbf{S}(\mathbf{x},t,\mathbf{A}) : \mathbf{A} \geq C|\mathbf{A}|^{q(\mathbf{x},t)}$ for all $\mathbf{A}$ in $\mathbb{M}^N_{\rm sym}$ and for a.a. $(\mathbf{x},t)$ in Q_T;
(D) monotonicity condition: $(\mathbf{S}(\mathbf{x},t,\mathbf{A}) - \mathbf{S}(\mathbf{x},t,\mathbf{B})) : (\mathbf{A}-\mathbf{B}) > 0$ for all $\mathbf{A}$ and $\mathbf{B}$ in $\mathbb{M}^N_{\rm sym}$, with $\mathbf{A} \neq \mathbf{B}$, and for a.a. $(\mathbf{x},t)$ in Q_T.

Here C denotes a positive constant which may vary from line to line and $\mathbb{M}^N_{\rm sym}$ is the vector space of all symmetric $N \times N$ matrices, which is equipped with the scalar product $\mathbf{A} : \mathbf{B}$ and norm $|\mathbf{A}| = \sqrt{\mathbf{A} : \mathbf{A}}$. Throughout this text we shall drop the dependence of $\mathbf{S}$ on $(\mathbf{x},t,\mathbf{A})$, writing only the different dependences on the tensor $\mathbf{A}$.

The best example of a fluid model obeying conditions (A)–(D), is a generalization of the well-known power-law model, widely used during the last century to describe diverse non-Newtonian fluids:

$$\mathbf{S} = \tau_0|\mathbf{D}|^{q(\cdot)-2}\mathbf{D}, \tag{1.6}$$

where q is the variable power-law index that characterizes the flow and τ_0 is the consistency factor, *i.e.* a positive constant that corresponds to the fluid viscosity when $q \equiv 2$. If q is constant and $1 < q < 2$, the fluid is pseudoplastic and it thins, *i.e.* it deforms more rapidly with an increase in shear stress, and therefore it is also called shear-thinning fluid. When $q = 2$, we recover the Stokes law and consequently the fluid is Newtonian. If $q > 2$, the fluid is dilatant and it thickens with an increase in shear stress, and due to that such fluids are also called shear-thickening. With slight modifications on the conditions (B)–(C), we can see that many other fluid models obey (A)–(D) as the Sisko and Carreau models (see *e.g.* [1]). Fluids modeled by the power-law (1.6) cannot be cataloged into a single class of non-Newtonian fluids as pseudo-plastic or dilatant. The problems we have in mind in this work are the cases when q depends on an electric field or a magnetic field, or still when q is a temperature-dependent function. Therefore the incompressible generalized Navier–Stokes equations (1.1)–(1.2) have to be solved with the Maxwell equations, in the cases of q depending on an electric or a magnetic field, or must be supplemented with the equation for the transport of the temperature, when q is temperature-dependent. Nevertheless, the resulting governing equations are essentially uncoupled, hence the Maxwell equations, or the temperature equation, can be solved first. The solution of electric or magnetic field, or the temperature, can then be considered as a known function, resulting that the original problem reduces to the problem of the incompressible generalized Navier–Stokes equations (1.1)–(1.2) with the deviatoric part of the Cauchy stress tensor given by (1.6), with

$$q(\cdot) = q(\mathbf{x},t) \quad \text{for } (\mathbf{x},t) \in Q_T. \tag{1.7}$$

In these cases are the electro, magneto or thermo-rheological fluids whose rheological properties are controllable through the application of an electric or magnetic field or by changes in the temperature, showing useful and special function with the effect of reversibility. Now we are in the presence of fluids that can go, for instance, from the consistency of a liquid (Newtonian) to that of a gel (non-Newtonian), and back, with response times on the order of milliseconds. Examples of electro-rheological fluids are suspensions dispersed with some polymeric colloids and typical magneto-rheological fluids are made of very small solid particles that are suspended in a Newtonian fluid. Applications of electro and magneto-rheological fluids in the automotive industry such as dampers, clutches, brakes and active bearings have already came to the market. Other application is seismic dampers, which are used in buildings in seismically-active zones to damp the oscillations (see *e.g.* Hao [2] and Henrie and Carlson [3]). Because electro and magneto-rheological fluids have the ability to change their flow characteristics according to the practical needs, they are often referred to as smart fluids. On the other hand, thermo-rheological fluids are made of nanometer-sized particles dispersed in a Newtonian fluid and are being used, for instance, to model certain cooling processes of volcano lava flows (see *e.g.* Das et al. [4]). Due to their trembling shear behavior, fluids with the deviatoric part of the stress tensor satisfying conditions (A)–(D), shall be called, in the sequel, by *trembling fluids*.

The mathematical analysis of non-Newtonian fluid models, with constant power-law indexes q, started with the works of Ladyzhenskaya [5,6] and then followed by Lions [7]. A few years ago the existence results established in [5–7] were improved by Zhikov [8], Wolf [9] and Diening et al. [10] for values of q that reach the pseudo-plastic zone. The asymptotic behavior of the solutions to these problems, with constant power-law indexes q, have been performed, among others, by Antontsev et al. [11] and Bae [12]. In recent years there has been a demand looking for similar results for the same problems, but considering variable power-law indexes q. The aim of this work is to study the asymptotic behavior of the weak solutions to the problem (1.1)–(1.4), with the deviatoric tensor $\mathbf{S}$ obeying conditions (A)–(D). We shall study the extinction in a finite time property and the large time behavior of the weak solutions. Perturbations of the asymptotically stable equilibrium will be analyzed as well.

The notation used throughout this article is largely standard in Mathematical Analysis and in particular in Mathematical Fluid Mechanics (see *e.g.* Lions [7]). We distinguish tensors and vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. The symbol C, with or without subscripts, will denote a generic positive constant, whose value will not be specified—it can change from one inequality to another. The dependence of C on other constants or parameters will always be clear from the exposition. Given $k \in \mathbb{N}$, we denote by $\mathrm{C}^k(\Omega)$ the space of all k-differentiable functions in Ω. By $\mathrm{C}^\infty_0(\Omega)$, we denote the space of all infinity-differentiable functions with compact support in Ω. If X is a generic Banach space, its dual space is denoted by X$'$. Let $1 \leq q \leq \infty$ and $\Omega \subset \mathbb{R}^N$, with $N \geq 1$, be a domain. We will use the classical Lebesgue spaces $\mathrm{L}^q(\Omega)$, whose norm is denoted by $\|\cdot\|_{\mathrm{L}^q(\Omega)}$. $\mathrm{W}^{1,q}(\Omega)$ denotes the Sobolev space of all functions $u \in \mathrm{L}^q(\Omega)$ such that the weak derivatives $\mathrm{D}^\gamma u$ exist, in the generalized sense, and are in $\mathrm{L}^q(\Omega)$